COMMON BOREL DIRECTIONS OF MEROMORPHIC FUNCTIONS AND THEIR DERIVATIVES

Yang Lo (杨 乐)
(*Institute of Mathematics, Academia Sinica*)

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Abstract

A general theorem on common filling regions of meromorphic functions and their derivatives is proved by a direct and simple method. Some important results whose original proofs are very long and complicated can be deduced immediately from this theorem.

For every meromorphic function of positive and finite order in the plane G. Valiron[1] proves that there exists at least a Borel direction. At the same time, he has posed an interesting and difficult problem: whether a meromorphic function and its derivatives have a common Borel direction or not. Concerning this problem, H. Milloux[2] has obtained the following theorem:

If $f(z)$ is an entire function of order λ $(0<\lambda<\infty)$, then every Borel direction of the derivative $f'(z)$ is also a Borel direction of $f(z)$.

The Milloux's proof is very long and complicated. (His paper is over eighty pages.) Recently K. H. Chang[3] has given a simpler proof for the Milloux theorem and extended it to the case of meromorphic functions having a Borel exceptional value ∞. However, the arrangement for original values in Chang's proof remains complicated.

In this paper we shall prove a general theorem, from which the Milloux's theorem and Chang's theorems can be obtained immediately. The proof of this general theorem is direct and simple.

I. Lemma

Let $f(z)$ be a meromorphic function in $|z|\leqslant R$ $(0<R<\infty)$. If $|z|\leqslant r$ $(0<r<R)$ and d is the distance of z from the nearest of the zeros and poles of $f(z)$, then

$$\log\left|\frac{f'(z)}{f(z)}\right|\leqslant\frac{R+r}{R-r}\,m\left(R,\frac{f'}{f}\right)+\{\bar{n}(R,\infty)+n(R,0)\}\left(\log\frac{1}{d}+\log 2R\right)-\frac{(R-r)^2}{4R^2}n(r,f'=0),\tag{1}$$

where $\bar{n}(R,\infty)$ denotes the number of reduced poles of $f(z)$ in $|z|\leqslant R$. (i.e. *every multiple pole is counted only once.*)

The Lemma can be proved by applying the Poisson-Jensen formula to $\frac{f'(z)}{f(z)}$. (See [4, 446—447].)

II. Theorem

Suppose that $f(z)$ is a meromorphic function of order λ $(0<\lambda<\infty)$ in the plane and that $f(z)$ adopts the infinity as a Borel exceptional value in $|\arg z|<\gamma_0$ $(\gamma_0>0)$. Let

$$\Gamma_n:\ |z-R_n|<\varepsilon_n R_n,\ R_{n+1}>2R_n,\ \lim_{n\to\infty}\varepsilon_n=0 \tag{2}$$

be a sequence of filling disks[1] of order λ of $f'(z)$. (That is to say, $f'(z)$ takes every complex number at least $R_n^{\lambda-\varepsilon'_n}$ times in Γ_n, except some numbers enclosed in two spherical circles with radii δ_n on the Riemann sphere, where $\lim\limits_{n\to\infty}\varepsilon'_n=\lim\limits_{n\to\infty}\delta_n=0$.) If we denote

$$\beta_n=\left(\sup_{r\geqslant R_n^{\frac{1}{2}}}\frac{\log T(r,f)}{\log r}\right)-\lambda \tag{3}$$

and

$$\varepsilon_n\geqslant\max\left(\frac{2\varepsilon'_n}{\lambda},\ \frac{2\beta_n}{\lambda},\ \frac{1}{(\log R_n)^{\frac{1}{2}}}\right), \tag{4}$$

then the regions

$$G_n:\ \left(\frac{R_n^{1-\eta_n}}{2}<|z|<2R_n^{1+\eta_n}\right)\cap(|\arg z|<20\pi\eta_n), \tag{5}$$

$$\eta_n=4\pi\varepsilon_n^{\frac{1}{2}}, \tag{6}$$

must contain a subsequence (G_{n_k}) as filling regions of order λ, i.e. *$f(z)$ takes every complex number at least $R_{n_k}^{\lambda-\varepsilon''_{n_k}}$ times in G_{n_k}, except some numbers enclosed in two spherical circles with radii δ'_{n_k} on the Riemann sphere, where $\lim\limits_{k\to\infty}\varepsilon''_{n_k}=\lim\limits_{k\to\infty}\delta'_{n_k}=0$.*

Proof. If the conclusion of the Theorem is not true, then any subsequence of filling regions can not be found from (G_n). We shall start from this fact and derive a contradiction.

Most of the inequalities in the present paper are only valid for sufficiently large values of the indice n. Hereinafter we shall not indicate this point.

Since (Γ_n) is a sequence of filling disks of $f'(z)$, there exists a number a_n such that[2]

$$0<|a_n|<1 \text{ and } n(\Gamma_n,\ f'=a_n)>R_n^{\lambda-\varepsilon'_n}. \tag{7}$$

In the interval $[R_n^{1-\eta_n}, R_n^{1+\eta_n}]$, we take the points

$$r'_{n,m}=R_n^{1-\eta_n}(1+\eta_n)^m,\quad \left(m=0,1,2,\cdots,M;\ M=\left[\frac{2\eta_n\log R_n}{\log(1+\eta_n)}\right]+1\right),$$

1) We use filling disks instead of the French term cercles de remplissage.

2) $n(D,\ g=\alpha)$ denotes the number of zeros of $g(z)-\alpha$ in D, counting with their multiplicities. When D is $|z-z_0|<r$, the notation $n(r,\ z_0,\ g=\alpha)$ is also used.

where $\left[\frac{2\eta_n \log R_n}{\log(1+\eta_n)}\right]$ denotes the integral part of $\frac{2\eta_n \log R_n}{\log(1+\eta_n)}$.

Put

$$C_{n,m}:\ |z - r'_{n,m}| < 2\eta_n r'_{n,m},$$
$$C'_{n,m}:\ |z - r'_{n,m}| < 40\eta_n r'_{n,m},$$

and

$$G'_n:\ (R_n^{1-\eta_n} < |z| < R_n^{1+\eta_n}) \cap (|\arg z| < \eta_n). \tag{8}$$

It is easy to see that

$$G'_n \subset \left(\bigcup_{m=0}^{M} C_{n,m}\right) \subset \left(\bigcup_{m=0}^{M} C'_{n,m}\right) \subset G_n. \tag{9}$$

Since (G_n) does not contain any subsequence as filling regions of order λ of $f(z)$, we can choose a subsequence (G_{n_k}) having the following properties:

For every positive integer k, there are three distinct complex numbers α_{i,n_k} $(i=1,2,3)$ such that $|\alpha_{i,n_k},\ \alpha_{j,n_k}| > \delta$ $(1 \leqslant i \neq j \leqslant 3)$ and $\sum_{i=1}^{3} n(G_{n_k}, f=\alpha_{i,n_k}) < R_{n_k}^{\rho_1}$, where δ and ρ_1 $(\rho_1 < \lambda)$ are two positive numbers independent of k.

In fact, we take two sequences of positive numbers ε''_k, δ'_k such that $\lim_{k\to\infty} \varepsilon''_k = \lim_{k\to\infty} \delta'_k = 0$. If the preceding assertion is not true, then a subsequence $(G_{n,1})$ of (G_n) can be found such that all the complex numbers satisfying the inequality $n(G_{n,1}, f = \alpha) < R_{n,1}^{\lambda-\varepsilon''_1}$ can be enclosed in two spherical circles with radii δ'_1 on the Riemann sphere. Similarly, there is a subsequence $(G_{n,2})$ of $(G_{n,1})$ such that all the complex numbers satisfying the inequality $n(G_{n,2},\ f = \alpha) < R_{n,2}^{\lambda-\varepsilon''_2}$ can be enclosed in two spherical circles with radii δ'_2. By continuing this procedure and taking the diagonal sequence $(G_{k,k})$, the complex numbers satisfying the inequality $n(G_{k,k},\ f = \alpha) < R_{k,k}^{\lambda-\varepsilon''_k}$ can be enclosed in two spherical circles with radii δ'_k, where $\lim_{k\to\infty} \varepsilon''_k = \lim_{k\to\infty} \delta'_k = 0$. This means $(G_{k,k})$ is a sequence of filling regions of order λ and we derive a contradiction.

In what follows we shall use (G_n) instead of (G_{n_k}) for the sake of brevity. It is obvious that we can take $\alpha_{3,n} = \infty$ $(n = 1, 2, \cdots)$. Hence, for every n, there are three distinct complex numbers $\alpha_{i,n}$ $(i = 1, 2, 3)$ such that

$$\alpha_{3,n} = \infty,\ \max\left\{|\alpha_{1,n}|,\ |\alpha_{2,n}|,\ \frac{1}{|\alpha_{1,n} - \alpha_{2,n}|}\right\} \leqslant \frac{2}{\delta},$$

and

$$\sum_{i=1}^{3} n(G_n,\ f = \alpha_{i,n}) < R_n^{\rho_1},$$

where δ and ρ_1 $(\rho_1 < \lambda)$ are two positive numbers independent of n.

By putting

$$h_n(z) = f(z) - a_n z$$

and

$$G_{n,m}(t) = h_n(r'_{n,m} + 40\eta_n r'_{n,m} t),$$

$G_{n,m}(t)$ is meromorphic in $|t| < 1$ and

$$\sum_{i=1}^{3} n(|t| < 1, G_{n,m}(t) = P_{i,n,m}(t)) < R_n^{\rho_1},$$

where $P_{i,n,m}(t) = \alpha_{i,n} - a_n r'_{n,m} - 40 a_n \eta_n r'_{n,m} t \quad (i = 1, 2, 3)$. The functions $P_{i,n,m}(t)$ have no zeros and poles in $|t| < 1$, and

$$\iint_{|t|<1} \log^+ \left(\sum_{i=1}^{2} |P_{i,n,m}(t)| + \sum_{1 \leqslant i \neq j \leqslant 3} \frac{1}{|P_{i,n,m}(t) - P_{j,n,m}(t)|} \right) d\sigma_t$$
$$= O(\log R_n). \tag{10}$$

According to the Rauch Theorem[1,p.21], the inequality $n\left(|t| < \frac{1}{20}, \ G_{n,m} = \alpha\right) < AR_n^{\rho_1}$ holds for all the complex numbers α, except some α enclosed in one spherical circle with radius $e^{-R_n^{\rho_1}}$. Thus, $n(C_{n,m}, \ h_n = \alpha) < AR_n^{\rho_1}$ holds for all the α, outside a spherical circle with radius $e^{-R_n^{\rho_1}}$.

Since $M \leqslant 4 \log R_n + 1$, there is a finite complex number b_n, outside the M exceptional circles with spherical radii $e^{-R_n^{\rho_1}}$ such that

$$|b_n| < 1, \quad |f(0) - b_n| > \frac{1}{2},$$
$$n(G'_n, h_n = b_n) < R_n^{\rho}, \ (\rho < \lambda). \tag{11}$$

Let

$$k_n = \frac{2\eta_n}{\pi} \tag{12}$$

and

$$\zeta = \zeta_n(z) = \frac{z^{\frac{1}{k_n}} - R_n^{\frac{1}{k_n}}}{z^{\frac{1}{k_n}} + R_n^{\frac{1}{k_n}}}. \tag{13}$$

Then the function $\zeta = \zeta_n(z)$ maps $|\arg z| < \eta_n$ to $|\zeta| < 1$. Its inverse is

$$z = z_n(\zeta) = R_n \left(\frac{1 + \zeta}{1 - \zeta} \right)^{k_n}, \tag{14}$$

and we denote $h_n(z_n(\zeta))$ by $H_n(\zeta)$.

When a point ζ is in $|\zeta| \leqslant 1 - \frac{2}{R_n^{\frac{\pi}{2}}}$, its original image z will satisfy

$$R_n^{1-\eta_n} \leqslant |z| \leqslant R_n^{1+\eta_n} \tag{15}$$

by (14) and (12). Since $f(z)$ adopts ∞ as a Borel exceptional value in $|\arg z| < \gamma_0$, (15), (8) and (11) imply

$$n\left(|\zeta| \leqslant 1 - \frac{2}{R_n^{\frac{\pi}{2}}},\ H_n = \infty\right) + n\left(|\zeta| \leqslant 1 - \frac{2}{R_n^{\frac{\pi}{2}}},\ H_n = b_n\right)$$
$$\leqslant n(G_n', h_n = \infty) + n(G_n', h_n = b_n) < R_n^{\rho'}, \quad (\rho' < \lambda). \tag{16}$$

Further, if ζ is the image of an arbitrary point $z = re^{i\theta} \in \Gamma_n$, then

$$|\zeta| = \left\{1 - \frac{4r^{\frac{1}{k_n}} R_n^{\frac{1}{k_n}} \cos\frac{\theta}{k_n}}{r^{\frac{2}{k_n}} + R_n^{\frac{2}{k_n}} + 2r^{\frac{1}{k_n}} R_n^{\frac{1}{k_n}} \cos\frac{\theta}{k_n}}\right\}^{\frac{1}{2}}$$
$$\leqslant \left\{1 - \frac{4(1-\varepsilon_n)^{\frac{1}{k_n}} \cos\frac{\theta}{k_n}}{[(1+\varepsilon_n)^{\frac{1}{k_n}} + 1]^2}\right\}^{\frac{1}{2}}. \tag{17}$$

(12) and (6) give

$$\frac{\varepsilon_n}{k_n} = \frac{\varepsilon_n^{\frac{1}{2}}}{8} \to 0.$$

Hence

$$\frac{\theta}{k_n} \leqslant \frac{\frac{\pi}{2}\varepsilon_n}{k_n} \to 0.$$

Since $(1-\varepsilon_n)^{\frac{1}{\varepsilon_n}} \to e^{-1}$, we have

$$(1-\varepsilon_n)^{\frac{1}{k_n}} = \{(1-\varepsilon_n)^{\frac{1}{\varepsilon_n}}\}^{\frac{\varepsilon_n}{k_n}} \to 1,$$

and

$$1 \leqslant (1+\varepsilon_n)^{\frac{1}{k_n}} \leqslant \frac{1}{(1-\varepsilon_n)^{\frac{1}{k_n}}} \to 1.$$

Therefore (17) means that the image of Γ_n under the mapping $\zeta = \zeta_n(z)$ is contained in $|\zeta| < \frac{1}{2}$.

Put

$$\tau_n = 8\varepsilon_n. \tag{18}$$

From (18), (12), (6) and (4), we deduce that

$$R_n^{\frac{\tau_n}{k_n}} \geqslant R_n^{\varepsilon_n^{\frac{1}{2}}} \geqslant e^{(\log R_n)^{\frac{3}{4}}} \to \infty. \tag{19}$$

Consequently, the image of Γ_n is contained in $|\zeta| < 1 - \frac{6}{R_n^{\frac{\tau_n}{k_n}}}$ and we have

$$n\left(1 - \frac{6}{R_n^{\frac{\tau_n}{k_n}}},\ H'_n = 0\right) \geqslant n(\Gamma_n, f' = a_n) > R_n^{\lambda - \varepsilon'_n} \tag{20}$$

by (7).

In $|\zeta| \leqslant 1 - \frac{2}{R_n^{\frac{\pi}{2}}}$, we make some disks, having their centers at every pole and b_n-point of $H_n(\zeta)$ and $d_n = \frac{1}{R_n^{\lambda+3}}$ for their radii. The union of these disks is denoted by $(\gamma)_{\zeta,n}$. Then we select $r_{1,n}$ and $r_{2,n}$ such that

$$r_{1,n} = 1 - \frac{6}{R_n^{\frac{\tau_n}{k_n}}}, \tag{21}$$

$$1 - \frac{4}{R_n^{\frac{\pi}{2}}} < r_{2,n} < 1 - \frac{3}{R_n^{\frac{\pi}{2}}}, \quad (|\zeta| = r_{2,n}) \cap (\gamma)_{\zeta,n} = \varnothing. \tag{22}$$

For any point ζ in the region $(|\zeta| \leqslant r_{1,n}) - (\gamma)_{\zeta,n}$, we apply the Lemma and obtain

$$\begin{aligned}\log\left|\frac{H'_n(\zeta)}{H_n(\zeta) - b_n}\right| &\leqslant \frac{r_{2,n} + r_{1,n}}{r_{2,n} - r_{1,n}}\, m\left(r_{2,n}, \frac{H'_n}{H_n - b_n}\right) \\ &+ \{\bar{n}(r_{2,n}, H_n = \infty) + n(r_{2,n}, H_n = b_n)\} \\ &\times \left(\log 2 + \log\frac{1}{d_n}\right) - \frac{(r_{2,n} - r_{1,n})^2}{4r_{2,n}^2}\, n(r_{1,n}, H'_n = 0).\end{aligned} \tag{23}$$

For the term $m\left(r_{2,n}, \frac{H'_n}{H_n - b_n}\right)$, we write

$$\begin{aligned} m\left(r_{2,n}, \frac{H'_n}{H_n - b_n}\right) &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left\{\left|\frac{h'_n(z_n(r_{2,n}e^{i\varphi}))}{h_n(z_n(r_{2,n}e^{i\varphi})) - b_n}\right| |z'_n(r_{2,n}e^{i\varphi})|\, d\varphi\right\} \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{h'_n(z_n(r_{2,n}e^{i\varphi}))}{h_n(z_n(r_{2,n}e^{i\varphi})) - b_n}\right| d\varphi + \frac{1}{2\pi}\int_0^{2\pi}\log^+|z'_n(r_{2,n}e^{i\varphi})|\, d\varphi. \end{aligned} \tag{24}$$

From (14), it is clear that

$$\frac{k_n R_n (1 - r_{2,n})^{k_n - 1}}{2^{k_n}} \leqslant |z'_n(r_{2,n}e^{i\varphi})| \leqslant \frac{2^{k_n} k_n R_n}{(1 - r_{2,n})^{k_n+1}}. \tag{25}$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+|z'_n(r_{2,n}e^{i\varphi})|\, d\varphi \leqslant \log^+ \frac{2^{k_n} k_n R_n}{(1 - r_{2,n})^{k_n+1}} \leqslant 3\log^+ R_n. \tag{26}$$

In order to estimate the integral $\frac{1}{2\pi}\int_0^{2\pi} \log^+\left|\frac{h'_n(z_n(r_{2,n}e^{i\varphi}))}{h_n(z_n(r_{2,n}e^{i\varphi})) - b_n}\right| d\varphi$, we recall the following fact[5, p.37]:

Suppose that $g(z)$ is meromorphic in $|z| \leqslant R$ ($\leqslant \infty$) and that $g(0) \neq 0, \infty$. Then we have

$$\log\left|\frac{g'(t)}{g(t)}\right| \leqslant 5 + 3\log^+\rho + 3\log^+\frac{1}{\rho - r} + \log^+\frac{\mathfrak{N}}{r} + \log^+\frac{1}{\delta(t)} + \log^+T(\rho, g) + \log^+\log^+\frac{1}{|g(0)|}, \tag{27}$$

for $t = re^{i\theta}$ and $0 < r < \rho < R$, where $\mathfrak{N} = n(\rho, g) + n\left(\rho, \frac{1}{g}\right)$ and $\delta(t)$ is the distance of t from the nearest of all the zeros and the poles of $g(z)$ in $|z| \leqslant \rho$.

When $g(0) = \infty$, set $g(z) = \frac{c_\lambda g_1(z)}{z^\lambda}$, where λ and c_λ are chosen such that $g_1(0) = 1$. From

$$\frac{g'(z)}{g(z)} = \frac{g_1'(z)}{g_1(z)} - \frac{\lambda}{z},$$

we have

$$\log^+\left|\frac{g'(t)}{g(t)}\right| \leqslant \log^+\left|\frac{g_1'(t)}{g_1(t)}\right| + \log^+\frac{\lambda}{r} + \log 2.$$

Thus

$$\log^+\left|\frac{g'(t)}{g(t)}\right| \leqslant 8 + 2\log\lambda + 4\log^+\rho + \log^+\frac{1}{r} + 3\log^+\frac{1}{\rho - r} + \log^+\frac{\mathfrak{N}}{r} + \log^+\frac{1}{\delta(t)} + \log^+T(\rho, g) + \log^+\log^+\frac{1}{|c_\lambda|}. \tag{27$'$}$$

Choose

$$g(z) = h_n(z) - b_n,\ t = z_n(r_{2,n}e^{i\varphi}),\ \rho = \frac{2^{k_n+1}R_n}{(1 - r_{2,n})^{k_n}}. \tag{28}$$

From (22) and

$$\frac{R_n(1 - r_{2,n})^{k_n}}{2^{k_n}} \leqslant |z_n(r_{2,n}e^{i\varphi})| \leqslant \frac{2^{k_n}R_n}{(1 - r_{2,n})^{k_n}},$$

we have

$$R_n^{1-\eta_n} < |t| = r < \rho < 2R_n^{1+\eta_n},\quad \rho - r \geqslant \frac{2^{k_n}R_n}{(1 - r_{2,n})^{k_n}} \geqslant R_n^{1-\eta_n}, \tag{29}$$

$$\mathfrak{N} = n\left(\frac{2^{k_n+1}R_n}{(1 - r_{2,n})^{k_n}},\ h_n\right) + n\left(\frac{2^{k_n+1}R_n}{(1 - r_{2,n})^{k_n}}, h_n = b_n\right) < R_n^{\lambda+1}, \tag{30}$$

$$T(\rho, g) = T\left(\frac{2^{k_n+1}R_n}{(1 - r_{2,n})^{k_n}}, f - a_n z - b_n\right) < R_n^{\lambda+1}, \tag{31}$$

$$|g(0)| = |f(0) - b_n| > \frac{1}{2}.\text{[1]} \tag{32}$$

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1) When $f(0) = \infty$, we note that $\lim_{z\to 0} f(z)z^\lambda = \lim_{z\to 0} g(z)z^\lambda = c_\lambda$ is a finite and non-zero number.

Now let us estimate the quantity $\delta(t)$. If $\zeta = re^{i\varphi}\left(\frac{1}{2} < r < 1\right)$ is a point in the ζ plane, then we have for its original image z

$$\begin{aligned}\arg z &= k_n \arg\frac{1+\zeta}{1-\zeta} = k_n \arcsin\frac{2r\sin\varphi}{\{(1-r^2)^2 + 4r^2\sin^2\varphi\}^{\frac{1}{2}}}\\ &\leqslant k_n\frac{\pi}{2}\cdot\frac{2r}{1+r^2} = \eta_n\frac{1}{1+\dfrac{(1-r)^2}{2r}}\\ &\leqslant \eta_n\left\{1-\frac{(1-r)^2}{4r}\right\}.\end{aligned}$$

In particular, for a point ζ on $|\zeta| = r_{2,n}$, its original image z must satisfy

$$\arg z \leqslant \eta_n\left\{1-\frac{\left(\dfrac{3}{R_n^{\frac{\pi}{2}}}\right)^2}{4\left(1-\dfrac{3}{R_n^{\frac{\pi}{2}}}\right)}\right\} \leqslant \eta_n\left(1-\frac{2}{R_n^{\pi}}\right). \tag{33}$$

If x_j is a pole or b_n-point of $h_n(z)$ in the region $\{(|z|\leqslant\rho)\backslash(|\arg z|<\eta_n)\}$, then

$$|t-x_j| \geqslant R_n^{1-\eta_n}\sin\frac{2\eta_n}{R_n^{\pi}} \geqslant \frac{1}{R_n^{\pi}} \tag{34}$$

by (29) and (33).

For an arbitrary point ζ in $|\zeta|\leqslant 1$, by analogy to the inequality (25), we obtain from (12), (6) and (4)

$$\begin{aligned}|z_n'(\zeta)| &\geqslant \frac{k_nR_n(1-|\zeta|)^{k_n-1}}{2^{k_n}} \geqslant \frac{k_nR_n}{2}\\ &= 4\varepsilon_n^{\frac{1}{2}}R_n \geqslant \frac{4R_n}{(\log R_n)^{\frac{1}{4}}} \geqslant 1.\end{aligned} \tag{35}$$

Suppose that x_j' is a pole or b_n-point of $h_n(z)$ in $|\arg z| < \eta_n$ and that its image $\zeta_n(x_j')$ is in $|\zeta| \leqslant 1 - \dfrac{2}{R_n^{\frac{\pi}{2}}}$. By (28), $r_{2,n}e^{i\varphi}$ is the image of t, so that

$$\begin{aligned}|r_{2,n}e^{i\varphi} - \zeta_n(x_j')| &= \left|\int_{\overline{tx_j'}} \zeta_n'(z)dz\right|\\ &\leqslant \left(\max_{z\in\overline{tx_j'}}|\zeta_n'(z)|\right)|t-x_j'| \leqslant \left(\max_{|\zeta|\leqslant 1}\frac{1}{|z_n'(\zeta)|}\right)|t-x_j'|\\ &= \frac{|t-x_j'|}{\min\limits_{|\zeta|\leqslant 1}|z_n'(\zeta)|} \leqslant |t-x_j'|.\end{aligned}$$

Since $(|\zeta|=r_{2,n})\cap(\gamma)_{\zeta,n}=\emptyset$ by (22), we obtain

$$|t-x_j'|\geqslant d_n=\frac{1}{R_n^{\lambda+3}}. \tag{36}$$

Suppose further that x_j'' is a pole or b_n-point of $h_n(z)$ in $|\arg z|<\eta_n$ and that its image $\zeta_n(x_j'')$ is out of $|\zeta|\leqslant 1-\frac{2}{R_n^{\frac{\pi}{2}}}$. We have as above

$$|r_{2,n}e^{i\varphi}-\zeta_n(x_j'')|\leqslant(\max_{z\in\overline{tx_j''}}|\zeta_n'(z)|)|t-x_j''|$$
$$\leqslant\frac{|t-x_j''|}{\min\limits_{|\zeta|\leqslant 1}|z_n'(\zeta)|}\leqslant|t-x_j''|,$$

so that

$$|t-x_j''|\geqslant\frac{1}{R_n^{\frac{\pi}{2}}}. \tag{37}$$

The inequalities (34), (36) and (37) give

$$\log\frac{1}{\delta(t)}=\max\left\{\log\frac{1}{|t-x_j|},\ \log\frac{1}{|t-x_j'|},\ \log\frac{1}{|t-x_j''|}\right\}=O(\log R_n). \tag{38}$$

By substituting the estimations (29), (30), (31), (32) and (38) in (27)[1], we obtain

$$\log^+\left|\frac{h_n'(z_n(r_{2,n}e^{i\varphi}))}{h_n(z_n(r_{2,n}e^{i\varphi}))-b_n}\right|=O(\log R_n). \tag{39}$$

Thus

$$m\left(r_{2,n},\ \frac{H_n'}{H_n-b_n}\right)=O(\log R_n) \tag{40}$$

by (24), (26) and (39).

From (16), (21), (22), (40) and

$$\frac{(r_{2,n}-r_{1,n})^2}{4r_{2,n}^2}n(r_{1,n},H_n'=0)\geqslant\left(\frac{1}{R_n^{\frac{\tau_n}{k_n}}}\right)^2 n(r_{1,n},\ H_n'=0)>R_n^{\lambda-\varepsilon_n'-\frac{2\tau_n}{k_n}}, \tag{41}$$

we have by (23)

$$\log\left|\frac{H_n'(\zeta)}{H_n(\zeta)-b_n}\right|<-\frac{1}{2}R_n^{\lambda-\varepsilon_n'-\frac{2\tau_n}{k_n}}, \tag{42}$$

where the point ζ is in $|\zeta|\leqslant r_{1,n}$, but out of $(\gamma)_{\zeta,n}$.

Return to the z plane and take

$$D_n:\ (R_n^{1-\frac{\tau_n}{4}}<|z|<R_n^{1+\frac{\tau_n}{4}})\cap(|\arg z|<\tau_n). \tag{43}$$

1) When $f(0)=\infty$, we use (27)' instead of (27).

For $z = re^{i\theta} \in D_n$, its image ζ has to satisfy

$$|\zeta| \leqslant \left\{1 - \frac{4r^{\frac{1}{k_n}} R_n^{\frac{1}{k_n}} \cos\frac{\theta}{k_n}}{(r^{\frac{1}{k_n}} + R_n^{\frac{1}{k_n}})^2}\right\}^{\frac{1}{2}} \leqslant 1 - \frac{(R_n^{1-\frac{\tau_n}{4}})^{\frac{1}{k_n}} R_n^{\frac{1}{k_n}}}{\{2(R_n^{1+\frac{\tau_n}{4}})^{\frac{1}{k_n}}\}^2} < r_{1,n}.$$

Denoting by $(\gamma)_{z,n}$ the original image of $(\gamma)_{\zeta,n}$, we obtain for $z \in (D_n \backslash (\gamma)_{z,n})$

$$\log\left|\frac{h_n'(z)}{h_n(z) - b}\right| = \log\left|\frac{H_n'(\zeta)}{H_n(\zeta) - b_n}\right| + \log\frac{1}{|z_n'(\zeta)|} < -\frac{R_n^{\lambda - \varepsilon_n' - \frac{2\tau_n}{k_n}}}{2}, \tag{44}$$

where $\log\frac{1}{|z_n'(\zeta)|} \leqslant 0$ by (35).

On the other hand, for an arbitrary point $z_{0,n} \in \{(D_n \cap (|z| \leqslant 2R_n^{1-\frac{\tau_n}{4}})) \backslash (\gamma)_{z,n}\}$, the Poisson-Jensen formula gives

$$\begin{aligned}\log|h_n(z_{0,n}) - b_n| < &\frac{3R_n^{1-\frac{\tau_n}{4}} + 2R_n^{1-\frac{\tau_n}{4}}}{3R_n^{1-\frac{\tau_n}{4}} - 2R_n^{1-\frac{\tau_n}{4}}} m(3R_n^{1-\frac{\tau_n}{4}}, h_n - b_n) \\ &+ \sum_{\mu} \log\left|\frac{(3R_n^{1-\frac{\tau_n}{4}})^2 - \bar{c}_\mu z_{0,n}}{3R_n^{1-\frac{\tau_n}{4}}(z_{0,n} - c_\mu)}\right|,\end{aligned} \tag{45}$$

where the c_μ's denote the poles of $h_n(z)$ in $|z| \leqslant 3R_n^{1-\frac{\tau_n}{4}}$.

If c_μ is out of $|\arg z| < \eta_n$, then we have

$$\begin{aligned}|z_{0,n} - c_\mu| &\geqslant R_n^{1-\frac{\tau_n}{4}} \sin(\eta_n - \tau_n) \geqslant \frac{\eta_n R_n^{1-\frac{\tau_n}{4}}}{\pi} \\ &\geqslant 4\varepsilon_n^{\frac{1}{2}} R_n^{1-\frac{\tau_n}{4}} \geqslant \frac{4R_n^{1-\frac{\tau_n}{4}}}{(\log R_n)^{\frac{1}{4}}} \geqslant 1,\end{aligned} \tag{46}$$

by (43), (18), (6) and (4).

If c_μ is in $|\arg z| < \eta_n$, its image ζ_μ must be in $|\zeta| \leqslant r_{1,n}$ since $|c_\mu| \leqslant 3R_n^{1-\frac{\tau_n}{4}}$. Denote by $\zeta_{0,n}$ the image of $z_{0,n}$. It is clear that $\zeta_{0,n}$ is out of $(\gamma)_{\zeta,n}$. Thus

$$\begin{aligned}d_n \leqslant |\zeta_{0,n} - \zeta_\mu| &= \left|\int_{\overline{z_{0,n}c_\mu}} \zeta_n'(z)\,dz\right| \leqslant \left(\max_{z \in \overline{z_{0,n}c_\mu}} |\zeta_n'(z)|\right)|z_{0,n} - c_\mu| \\ &\leqslant \left(\max_{|\zeta| \leqslant 1} \frac{1}{|z_n'(\zeta)|}\right)|z_{0,n} - c_\mu| = \frac{|z_{0,n} - c_\mu|}{\min_{|\zeta| \leqslant 1}|z_n'(\zeta)|} \leqslant |z_{0,n} - c_\mu|.\end{aligned} \tag{47}$$

By substituting (46) and (47) in (45),we have

$$\log|h_n(z_{0,n})-b_n|<5m(3R_n^{1-\frac{\tau_n}{4}},\ h_n-b_n)+n(3R_n^{1-\frac{\tau_n}{4}},\ h_n=\infty)\log\frac{6R_n^{1-\frac{\tau_n}{4}}}{d_n}$$

$$<\left(5+\frac{\log\dfrac{6R_n^{1-\frac{\tau_n}{4}}}{d_n}}{\log\dfrac{4}{3}}\right)T(4R_n^{1-\frac{\tau_n}{4}},\ h_n-b_n).$$

From $h_n(z)=f(z)-a_nz$, (7), (11), (18), (3) and (4), we obtain

$$\begin{aligned}\log|h_n(z_{0,n})-b_n|&<(\lambda+5)(\log R_n)T(4R_n^{1-2\varepsilon_n},\ f)\\&<4^{\lambda+1}(\lambda+5)(\log R_n)R_n^{\lambda-2\lambda\varepsilon_n+\beta_n-2\varepsilon_n\beta_n}\\&<R_n^{\lambda-\lambda\varepsilon_n}.\end{aligned}\tag{48}$$

Every contour of $(\gamma)_{z,n}$ can be covered by a corresponding disk with radius d_n'. The union of these disks will be denoted by $(\gamma)_{z,n}'$. It is easy to see that

$$d_n'\leqslant\left(\max_{|\zeta|\leqslant1-\frac{2}{R_n^{\frac{\pi}{2}}}}|z_n'(\zeta)|\right)d_n\leqslant\frac{2^{k_n}k_nR_n}{\left(\dfrac{2}{R_n^{\frac{\pi}{2}}}\right)^{k_n+1}}\cdot\frac{1}{R_n^{\lambda+3}}\leqslant\frac{1}{R_n^{\lambda+\frac{1}{4}}}.$$

In view of (16), the total sum of the radii of $(\gamma)_{z,n}'$ does not exceed

$$\left\{n\left(|\zeta|\leqslant1-\frac{2}{R_n^{\frac{\pi}{2}}},\ H_n=\infty\right)+n\left(|\zeta|\leqslant1-\frac{2}{R_n^{\frac{\pi}{2}}},\ H_n=b_n\right)\right\}d_n'<\frac{1}{R_n^{\frac{1}{4}}}.\tag{49}$$

For an arbitrary point z in $D_n\backslash(\gamma)_{z,n}'$, we may join it to the point $z_{0,n}$ with a segment. If the intersection parts of this segment with $(\gamma)_{z,n}'$ are replaced by the corresponding arcs, then we obtain a curve L_n. By (43) and (49), the length of L_n does not exceed $2R_n^{1+\frac{\tau_n}{4}}$. Thus

$$\left|\log\frac{h_n(z)-b_n}{h_n(z_{0,n})-b_n}\right|=\left|\int_{L_n}\frac{h_n'(u)}{h_n(u)-b_n}du\right|<e^{-\frac{1}{2}R_n^{\lambda-\varepsilon_n'-2\frac{\tau_n}{k_n}}}(2\pi+1)R_n^{1+\frac{\tau_n}{4}}<1.\tag{50}$$

Consequently

$$\log|h_n(z)-b_n|<\log|h_n(z_{0,n})-b_n|+1<R_n^{\lambda-\lambda\varepsilon_n}+1.\tag{51}$$

Combining this inequality with (44), we obtain

$$\log|h_n'(z)|<R_n^{\lambda-\lambda\varepsilon_n}+1-\frac{1}{2}R_n^{\lambda-\varepsilon_n'-\frac{2\tau_n}{k_n}}\tag{52}$$

for $z\in(D_n\backslash(\gamma)_{z,n}')$.

Now we choose a point z_n in D_n such that $|z_n-R_n|<1$ and $z_n\bar{\in}(\gamma)_{z,n}'$. Obviously, D_n contains the disk $|z-z_n|<4\varepsilon_nR_n$. In the annulus $3\varepsilon_nR_n<|z-z_n|<4\varepsilon_nR_n$, we choose a circumference $|z-z_n|=r_n$, not intersecting $(\gamma)_{z,n}'$. In view of (49), the above two choices are possible.

According to (52) and (7), we have

$$\log^+|f'(z)| \leqslant \log^+|h_n'(z)| + \log^+|a_n| + \log 2 < R_n^{\lambda-\lambda\varepsilon_n} \tag{53}$$

for every point z on $|z - z_n| = r_n$. It follows that

$$m(r_n, z_n, f') < R_n^{\lambda-\lambda\varepsilon_n}. \tag{54}$$

In the angular domain $|\arg z| < \gamma_0$, $f'(z)$ adopts ∞ as a Borel exceptional value, i. e.

$$n(r_n, z_n, f') < R_n^{\rho_1}, \quad (\rho_1 < \lambda).$$

If c_μ is an arbitrary pole of $f'(z)$ in $|z - z_n| < r_n$, then we have $|c_\mu - z_n| \geqslant d_n$, similar to the inequality (47). Thus

$$N(r_n, z_n, f') \leqslant \int_{d_n}^{r_n} \frac{n(t, z_n, f')}{t} dt < R_n^\rho, \quad (\rho < \lambda). \tag{55}$$

Therefore

$$T(r_n, z_n, f') < 2R_n^{\lambda-\lambda\varepsilon_n}. \tag{56}$$

On the other hand, we have for any complex number α

$$n(\Gamma_n, f' = \alpha) \leqslant n\left(\frac{3}{2}\varepsilon_n R_n, z_n, f' = \alpha\right) \leqslant \frac{1}{\log 2} N(r_n, z_n, f' - \alpha)$$

$$\leqslant \frac{1}{\log 2}\left\{T(r_n, z_n, f') + \log^+|\alpha| + \log\frac{1}{|f'(z_n) - \alpha|} + \log 2\right\}$$

$$\leqslant \frac{1}{\log 2}\left\{T(r_n, z_n, f') + \log^+\frac{1}{|f'(z_n), \alpha|} + \log 2\right\},$$

where $|f'(z_n), \alpha|$ denotes the spherical distance between $f'(z_n)$ and α. Substituting (56) in this inequality, we obtain

$$n(\Gamma_n, f' = \alpha) < \frac{3}{\log 2} R_n^{\lambda-\lambda\varepsilon_n}, \tag{57}$$

except some α enclosed in a spherical circle with radius $e^{-R_n^{\frac{\lambda}{2}}}$. But according to the supposition of the Theorem, (Γ_n) is a sequence of filling disks of order λ of $f'(z)$, so that

$$n(\Gamma_n, f' = \alpha) > R_n^{\lambda-\varepsilon_n'} \tag{58}$$

for all the complex numbers α, except some α in two spherical circles with radii δ_n.

Comparing (57) with (58), we derive $R_n^{\lambda\varepsilon_n - \varepsilon_n'} < \frac{3}{\log 2}$. But (4) implies $R_n^{\lambda\varepsilon_n - \varepsilon_n'} \geqslant R_n^{\frac{\lambda\varepsilon_n}{2}} \geqslant e^{\frac{\lambda}{2}(\log R_n)^{\frac{1}{2}}} \to \infty$. This contradiction completes the proof of the Theorem.

III. Corollaries

From the above general theorem, we can obtain four corollaries immediately. Among them, Corollaries 2 and 3 are Chang's results[3], which extend Milloux's theorems[2].

Corollary 1. *Let $f(z)$ be a meromorphic function of order λ $(0<\lambda<\infty)$ in the plane. Suppose that B: $\arg z=\theta_0$ $(0\leqslant\theta_0<2\pi)$ is a Borel direction of order λ of $f'(z)$ and that $f(z)$ adopts ∞ as a Borel exceptional value in $|\arg z-\theta_0|<\gamma_0$ $(\gamma_0>0)$. Then there exists a sequence of positive numbers R_{n_k} tending to ∞ and a sequence of positive numbers η_{n_k} tending to 0 such that*

$$\left(\frac{R_{n_k}^{1-\eta_{n_k}}}{2}<|z|<2R_{n_k}^{1+\eta_{n_k}}\right)\cap(|\arg z-\theta_0|<\eta_{n_k})$$

is a sequence of filling regions both for $f(z)$ and $f'(z)$.

Without loss of generality we can suppose that $\theta_0=0$. Since B: $\arg z=0$ is a Borel direction of order λ of $f'(z)$, according to the Rauch Theorem[1,p.33], there exists a sequence of filling disks of order λ, Γ_n^*: $|z-z_n|<\varepsilon_n^*|z_n|$, $|z_{n+1}|>2|z_n|$, $\lim\limits_{n\to\infty}\varepsilon_n^*=0$, $\lim\limits_{n\to\infty}\arg z_n=0$ such that $f'(z)$ takes every complex number α at least $|z_n|^{\lambda-\varepsilon'_n}$ times in Γ_n^*, except some numbers enclosed in two spherical circles with radii δ_n on the Riemann sphere, where $\lim\limits_{n\to\infty}\varepsilon'_n=\lim\limits_{n\to\infty}\delta_n=0$.

Choose

$$\varepsilon_n=\max\left\{\varepsilon_n^*+\arg z_n,\ \frac{2\varepsilon'_n}{\lambda},\ \frac{2\beta_n}{\lambda},\ \frac{1}{(\log R_n)^{\frac{1}{2}}}\right\},\tag{59}$$

where $R_n=|z_n|$ and β_n are given by (3). It is obvious that every Γ_n: $|z-R_n|<\varepsilon_nR_n$ contains the corresponding disk Γ_n^*. Thus (Γ_n) is a sequence of filling disks of order λ of $f'(z)$ and satisfies the conditions of the above theorem. On putting $\eta_n=4\pi\varepsilon_n^{\frac{1}{2}}$, then $\left(\frac{R_n^{1-\eta_n}}{2}<|z|<2R_n^{1+\eta_n}\right)\cap(|\arg z|<\eta_n)$ $(n=1,2,\cdots)$ must contain a subsequence of filling regions both for $f(z)$ and $f'(z)$.

Corollary 2. *With the supposition of the Corollary 1, B is a Borel direction of order λ of $f(z)$.*

In fact, from the Corollary 1, G_{n_k}: $\left(\frac{R_{n_k}^{1-\eta_{n_k}}}{2}<|z|<2R_{n_k}^{1+\eta_{n_k}}\right)\cap(|\arg z-\theta_0|<\eta_{n_k})$ is a sequence of filling regions of order λ of $f(z)$, i.e. $f(z)$ takes all the complex numbers α at least $R_{n_k}^{\lambda-\varepsilon''_{n_k}}$ times, except some numbers enclosed in two spherical circles with radii δ'_{n_k} on the Riemann sphere, where $\lim\limits_{k\to\infty}\varepsilon''_{n_k}=\lim\limits_{k\to\infty}\delta'_{n_k}=0$. We can suppose without loss of generality that $\sum\limits_{k=1}^{\infty}\delta'_{n_k}$ is less than a predeterminate positive number τ_0.

Consequently, the inequality $n(G_{n_k},f=\alpha)>R_{n_k}^{\lambda-\varepsilon''_{n_k}}$ holds for all the positive integers k and all the complex numbers α, except some α enclosed in a sequence of circles, and the total sum of their radii is less than r_0. For the "normal" numbers α and any positive number ε, we have

$$\lambda \geqslant \overline{\lim_{r\to\infty}} \frac{\log n(r, \theta_0, \varepsilon, f = \alpha)}{\log r} \geqslant \overline{\lim_{k\to\infty}} \frac{\log n(2R_{n_k}^{1+\eta_{n_k}}, \theta_0, \varepsilon, f = \alpha)}{\log (2R_{n_k}^{1+\eta_{n_k}})}$$

$$\geqslant \overline{\lim_{k\to\infty}} \frac{\log R_{n_k}^{\lambda-\varepsilon''_{n_k}}}{(1+\eta_{n_k})\log (2R_{n_k})} = \lambda.$$

Therefore

$$\lim_{\varepsilon\to 0} \left\{ \overline{\lim_{r\to\infty}} \frac{\log n(r, \theta_0, \varepsilon, f = \alpha)}{\log r} \right\} = \lambda \tag{60}$$

for all the "normal" numbers α. But a classical result of Valiron[1,p.32] says that if the set of complex numbers α satisfying the equality (60) has a positive measure, then $\arg z = \theta_0$ must be a Borel direction of order λ of $f(z)$. This gives the conclusion of Corollary 2.

Corollary 3. *Suppose that $f(z)$ is a meromorphic function of order λ $(0 < \lambda < \infty)$ in the plane and that $f(z)$ adopts ∞ as a Borel exceptional value. There exists at least a common Borel direction for $f(z)$ and all its derivatives.*

Corollary 4. *Suppose that $f(z)$ is a meromorphic function of order λ $\left(\frac{1}{2} < \lambda < \infty\right)$ in the plane and that $f(z)$ adopts ∞ as a Borel exceptional value. If $f(z)$ has exactly two Borel directions B_1 and B_2, then every $f^{(l)}(z)$ $(l = 1, 2, \cdots)$ takes exactly B_1 and B_2 as its Borel directions too.*

References

[1] Valiron, G.: *Directions de Borel des fonctions méromorphes*, Gauthier-Villars, Paris, (1938).

[2] Milloux, H.: Sur les directions de Borel des fonctions entières de leurs dérivées et de leurs integrales, *J. d'Analyse Math.*, **1** (1951), 244—330.

[3] 张广厚: 关于亚纯函数与其各级导数成积分的公共波莱耳方向的研究 (I), 数学学报, **20** (1977), 73—98.

[4] Yang Lo et Chang Kuan-heo: Sur la construction des fonctions méromorphes ayant des directions singulières données, *Sci. Sinica*, **19** (1976), 445—459.

[5] Hayman, W. K.: *Meromorphic Functions*, Oxford Math. Monographs, Oxford University Press, (1964).