GROWTH AND VALUES OF FUNCTIONS REGULAR IN AN ANGLE

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[Received 22 November 1979—Revised 19 June 1980]

1. *Introduction*

We consider a function $f(z)$ regular in an angle

(1.1) $$S = S(\alpha, \beta) = \{z \mid \alpha \leqslant \arg z \leqslant \beta, \, |z| > 0\},$$

and suppose that for some positive $\lambda, f(z^{\lambda})$ is regular† at $z = 0$. In the sequel we always include this second condition, when we say that f is regular in S. We write

(1.2) $$S' = S(\alpha', \beta'), \quad \text{where } \alpha < \alpha' < \beta' < \beta.$$

Let $n(r, a, S)$ be the number of a-points, i.e. roots of the equation $f(z) = a$ in the sector

(1.3) $$S(r) = \{z \mid z \in S \text{ and } |z| < r\}.$$

Our hypotheses ensure that $n(r, a, S)$ is finite. We also write

(1.4) $$M(r, S) = \sup_{\alpha \leqslant \theta \leqslant \beta} |f(re^{i\theta})|.$$

The order k of f in S is defined by

(1.5) $$k(S) = \limsup_{r \to \infty} \frac{\log^+ \log^+ M(r, S)}{\log r}.$$

The order of the a-points of f is defined by

(1.6) $$k(a, S) = \limsup_{r \to \infty} \frac{\log n(r, a, S)}{\log r}.$$

The following result is classical and due to Nevanlinna [3].

THEOREM A. *If for some S' we have*

(1.7) $$k(S') > \pi/(\beta - \alpha),$$

then we have for every a with at most one exception

(1.8) $$k(a, S) \geqslant k(S').$$

It is essential that strict inequality holds in the hypothesis (1.7). Thus if $f(z) = ze^z$, S is the angle $|\arg z| \leqslant \frac{1}{2}\pi$, and S' an angle $|\arg z| \leqslant \delta$, for $0 < \delta < \frac{1}{2}\pi$, then $k(S') = 1$, but

(1.9) $$f(z) \to \infty, \quad \text{as } z \to \infty \text{ in } S,$$

† This condition ensures that roots of $f(z) = a$ cannot accumulate at the origin, so that $n(r, a, S)$ is always finite.

Proc. London Math. Soc. (3), 44 (1982), 193–214.
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so that the equation $f(z) = a$ has at most a finite number of roots in S for every a, and so $k(a, S) = 0$ for every a.

M. L. Cartwright in looking through Littlewood's papers in the library of Trinity College, Cambridge recently discovered that Littlewood had in about 1930 been concerned with the question of finding some extension of Theorem A to the case when (1.7) is false. Clearly additional conditions will be needed to prevent (1.9). If merely $\beta-\alpha > 0$ Littlewood put forward the following

CONJECTURE. *If for some positive ρ we have*

$$k(S') \geqslant \rho \quad \text{and} \quad k(0, S') \geqslant \rho, \tag{1.10}$$

then $k(a, S) \geqslant \rho$ for every a with at most one exception or at least for most values.

2. *Statement of results*

It turns out that the hypotheses in the conjecture are still too weak. To achieve the desired conclusion it seems to be necessary to prove that $n(r, 0, S')$ and $M(r, S')$ are large for the same value of r, or that $f(z)$ has many zeros near some point in S where the function is large. This is not a consequence of (1.10). We can achieve our aim by assuming that f has lower order at least ρ in S' or at least that $M(r, S')$ is large for a fairly dense sequence of r. More precisely, we have

THEOREM 1. *Suppose that r_ν is an increasing sequence of positive numbers such that*

$$r_\nu \to \infty \tag{2.1}$$

and

$$\frac{\log r_{\nu+1}}{\log r_\nu} \to 1, \quad \text{as } \nu \to \infty. \tag{2.2}$$

Suppose further that

$$\liminf_{\nu \to \infty} \frac{\log \log M(r_\nu, S')}{\log r_\nu} \geqslant \rho > 0. \tag{2.3}$$

Then if $k(0, S') \geqslant \rho$, we have $k(a, S) \geqslant \rho$ for every a with at most one exception.

The exceptional value of a is essential in Theorem A and Theorem 1. Thus $f(z) = 1+e^z$ satisfies the conditions of both Theorems with $\rho = 1$, if $\beta-\alpha > \pi$, but $f(z) \neq 1$, so that $k(1, S) = 0$. Also the condition (2.2) is essential for the proof of Theorem 1. We prove

THEOREM 2. *Suppose that we are given δ, η, ρ such that $0 < \delta < \frac{1}{2}\pi$,*

$$1 < \eta < 2, \tag{2.4}$$

$$\tfrac{2}{3} < \rho < 1, \tag{2.5}$$

and a sequence r_ν tending to ∞ with ν. There exists an entire function $f(z)$ of order 1, mean type, such that if $S = S(-\frac{1}{2}\pi+\frac{3}{4}\delta, \frac{1}{2}\pi-\frac{3}{4}\delta)$, and $S' = S(-\frac{1}{2}\pi+\delta, \frac{1}{2}\pi-\delta)$, then $f(z)$ has order ρ in S and

$$\liminf \frac{\log |f(z)|}{|z|^\rho} > 0 \tag{2.6}$$

as $z \to \infty$ *in* S *outside the sequence of annuli*

$$r_\nu < |z| < r_\nu^\eta, \quad \text{for } \nu = 1, 2, \dots. \tag{2.7}$$

Further, $k(0, S') = \rho$, *but for* $a \neq 0$,

$$k(a, S) \leqslant \rho' = \rho - \frac{(1-\rho)(\eta-1)}{5}. \tag{2.8}$$

We note that δ, $1-\rho$, $\eta-1$ can be chosen as small as we please so that the openings of S, S' differ arbitrarily little from π/ρ. Further, (2.6) shows that $f(z)$ is of lower order at least ρ in S outside the sequence of annuli (2.7), only just bigger than those permitted by (2.2). In particular, Theorem 1 fails if r_ν is any increasing sequence not satisfying (2.2), for in this case there is a subsequence $\nu = \nu_n$ of n for which $r_{\nu+1} > r_\nu^\eta$, where $\eta > 1$, and we can apply Theorem 2 with r_{ν_n} instead of r_ν. Also by considering $f(z^\lambda e^{i\theta})$ instead of $f(z)$ we can obtain corresponding results in any angle $S(\alpha, \beta)$ (though our example will then no longer be entire in general).

3. *Proof of Theorem* 1: *preliminary results*

In order to prove Theorem 1, we need an improved form of Schottky's theorem [**2**, p. 129].

LEMMA 1. *Suppose that* $F(z)$ *is meromorphic in* $|z| < 1$ *and that the equations* $F(z) = 0, \infty, 1$ *have there at most a finite number of roots* a_λ *where* $\lambda = 1$ *to* L, b_μ *where* $\mu = 1$ *to* M, *and* c_ν *where* $\nu = 1$ *to* N, *respectively. We write*

$$F_0(z) = F(z) \prod_{\mu=1}^{M} \frac{z - b_\mu}{1 - \bar{b}_\mu z} \Big/ \prod_{\lambda=1}^{L} \left(\frac{z - a_\lambda}{1 - \bar{a}_\lambda z} \right). \tag{3.1}$$

Then if $z_1 = re^{i\theta}$, *where* $0 < r < 1$, $0 \leqslant \theta \leqslant 2\pi$, *and* $z_0 = 0$, *we have*

$$\log^+ |F_0(z_1)| \leqslant \frac{A_0}{1-r} (\log^+ |F_0(z_0)| + L + M + N + 1), \tag{3.2}$$

where A_0 *is a positive absolute constant.*

We deduce

LEMMA 2. *The conclusion* (3.2) *holds for an arbitrary pair of points* z_0, z_1 *in* $|z| < 1$, *with*

$$r = \left| \frac{z_1 - z_0}{1 - \bar{z}_0 z_1} \right|.$$

To prove Lemma 2 we simply apply Lemma 1 with

$$F\left(\frac{z_0 + z}{1 + \bar{z}_0 z}\right), \quad F_0\left(\frac{z_0 + z}{1 + \bar{z}_0 z}\right), \quad \frac{z_1 - z_0}{1 - \bar{z}_0 z_1}$$

instead of

$$F(z), \qquad F_0(z), \qquad z_1.$$

From now on we assume that $F(z)$ satisfies the hypotheses of Lemma 1 and is regular so that $M = 0$.

LEMMA 3. *Suppose that $0 < t < 1$ and that $C > 1$. Then if A_0 is as in* (3.2) *and*

$$\log|F(0)| > \frac{A_1}{1-t}\{\log C + L + N + 1\}, \quad \text{where } A_1 = 22A_0, \tag{3.3}$$

there exists r, such that $t < r < \frac{1}{2}(1+t)$ and

$$|F(re^{i\theta})| > C, \quad \text{for } 0 \leqslant \theta \leqslant 2\pi. \tag{3.4}$$

Hence for $|a| < C$, the equations $F(z) = a$ all have the same number of roots in $|z| < r$, that is, at most L. Also, if $\delta > 0$,

$$\log|F(z)| > \log C - L\log(4e/\delta), \quad \text{for } |z| < r, \tag{3.5}$$

outside a set of circles the sum of whose radii is at most δ.

We apply Lemma 2 with $z_1 = 0$ and $z_0 = re^{i\theta}$. This gives

$$\log^+|F_0(z_0)| \geqslant \frac{(1-r)}{A_0}\log^+|F_0(z_1)| - L - N - 1. \tag{3.6}$$

Next we have

$$\begin{aligned} \log|F(z_0)| &= \log|F_0(z_0)| + \sum \log\left|\frac{z_0 - a_\lambda}{1-\bar{a}_\lambda z_0}\right| \\ &\geqslant \log|F_0(z_0)| + \sum\log\left|\frac{r-|a_\lambda|}{1-|a_\lambda|r}\right|. \end{aligned} \tag{3.7}$$

Also for $0 < a < 1$ and $0 < r_1 < 1$, we have

$$\int_{r_1}^{1} \log\left|\frac{1-ar}{r-a}\right| dr < 5(1-r_1). \tag{3.8}$$

To see this we note that

$$\begin{aligned} \frac{1}{1-r_1}\int_{r_1}^{1}\log\left|\frac{1-ar}{r-a}\right| dr &< 2\int_{r_1}^{1}\log\left|\frac{1-ar}{r-a}\right|\frac{dr}{1-r^2} \\ &< 2\int_{-1}^{1}\log\left|\frac{1-ar}{r-a}\right|\frac{dr}{1-r^2} \\ &= 2\int_{-1}^{1}\log\left(\frac{1}{x}\right)\frac{dx}{1-x^2} \\ &= 4\int_{0}^{1}\log\left|\frac{1}{x}\right|\frac{dx}{1-x^2} \\ &= 4\sum_{n=0}^{\infty}\int_0^1 x^{2n}\log\left(\frac{1}{x}\right)dx \\ &= 4\sum_{n=0}^{\infty}\left(\frac{1}{2n+1}\right)^2 \\ &= \tfrac{1}{2}\pi^2 < 5. \end{aligned}$$

This proves (3.8).

We deduce that if $r_2 = \frac{1}{2}(1+r_1)$, then

$$\int_{r_1}^{r_2} \sum_{\lambda=1}^{L} \log\left|\frac{r-|a_\lambda|}{1-|a_\lambda|r}\right| dr > -5L(1-r_1).$$

Thus there exists r, such that $r_1 < r < r_2$ and

$$\sum_{\lambda=1}^{L} \log\left|\frac{r-|a_\lambda|}{1-|a_\lambda|r}\right| > \frac{-5L(1-r_1)}{r_2-r_1} = -10L.$$

We choose $r_1 = t$ and $r_2 = \frac{1}{2}(1+t)$ and take this value of r. Then (3.6) and (3.7) show that for $|z| = r$ we have

$$\begin{aligned}\log^+|F(z)| &\geqslant \log^+|F_0(z)| - 10L \\ &\geqslant \frac{1-r}{A_0}\log^+|F_0(0)| - L - N - 1 - 10L \\ &\geqslant \frac{1-t}{2A_0}\log^+|F(0)| - 11(L+N+1) > \log C,\end{aligned}$$

if

$$\log|F(0)| > \frac{2A_0}{1-t}\{\log C + 11(L+N+1)\}.$$

Taking $A_1 = 22A_0$, we deduce (3.4). To deduce the next statement we apply Rouché's theorem.

Next we consider

$$F_0(z) = F(z)\prod_{\lambda=1}^{L}\left(\frac{1-\bar{a}_\lambda z}{z-a_\lambda}\right),$$

and deduce that $|F_0(z)| > C$ for $|z| = r$ and hence for $|z| < r$, since $F_0(z) \neq 0$ in $|z| < r$. Thus

$$\begin{aligned}\log|F(z)| &= \log|F_0(z)| + \sum \log\left|\frac{z-a_\lambda}{1-\bar{a}_\lambda z}\right| \\ &> \log C - \sum\log|1-\bar{a}_\lambda z| + \sum\log|z-a_\lambda| \\ &> \log C - L\log 2 + \sum_{\lambda=1}^{L}\log|z-a_\lambda| \\ &> \log C - L(\log 2 + \log 1/h),\end{aligned}$$

outside a set of circles, the sum of whose radii is at most $2eh$, by Cartan's lemma [**1**, p. 46]. Writing $\delta = 2eh$, we obtain (3.5).

3.1. *Hyperbolic distances*

It is convenient to formulate (3.4) for an arbitrary simply connected domain and for this we use hyperbolic distance [**4**, p. 47 et seq.]. Suppose that D is a simply connected domain, and that w_1, w_2 are two points of D. Then the hyperbolic distance $d(w_1, w_2;\, D)$ is defined by

$$d(w_1, w_2; D) = \tfrac{1}{2}\log\frac{1+r}{1-r},$$

where D is mapped conformally onto $|z| < 1$ so that w_1, w_2 correspond to $z = 0, r$. Our reformulation of Lemma 3 is

LEMMA 4. *Suppose that $F(z)$ is regular in a simply connected domain D, and that the equations $F(z) = 0, 1$ have respectively L and N roots in D, where L, N are finite. Suppose that $C > 1$ and that w_1 is a point of D, such that*

$$\log|F(w_1)| > A_1 e^{2d}\{\log C + L + N + 1\},$$

where $d > 0$. Then for $|a| < C$, the equation $F(z) = a$ has at most L roots in the subdomain D' of points w of D, such that

$$d(w_1, w; D) < d.$$

Lemma 4 is a consequence of the first parts of Lemma 3 and conformal mapping. Finally, we need a lemma on hyperbolic distances.

LEMMA 5. *Suppose that D is the domain given by*

$$|\arg z| < \alpha, \quad R/K < |z| < KR,$$

where $K > e^{\alpha}$. Let D' be the subdomain of D given by

$$|\arg z| < \alpha(1-\delta), \quad e^{\alpha}R/K < |z| < KR/e^{\alpha},$$

where δ is a positive number. Let w_1 be a point of D' on $|z| = R$, and let w_2 be any other point of D'. Then

$$d(w_1, w_2; D) < \frac{\pi}{4\alpha}\log K + \log(3/\delta).$$

We assume first that

$$R = 1, \quad \alpha = \tfrac{1}{2}\pi, \quad w_1 = 1, \quad w_2 = r,$$

where $2/K < r < 1$. Then D contains the disk

$$D_0\colon |w-1| < 1 - K^{-1}.$$

Since hyperbolic distances decrease with expanding domain we deduce that

$$\begin{aligned} d(w_1, w_2; D) < d(w_1, w_2; D_0) &= \tfrac{1}{2}\log\frac{1+|w_2-w_1|/(1-K^{-1})}{1-|w_2-w_1|/(1-K^{-1})} \\ &< \tfrac{1}{2}\log\left\{\frac{2}{1-(1-2/K)/(1-1/K)}\right\} \\ &= \tfrac{1}{2}\log\{2(K-1)\} < \tfrac{1}{2}\log K + \tfrac{1}{2}\log 2. \end{aligned}$$

Next if $1 < r < K/2$, we use the transformation $w' = 1/w$ which maps the domains D, D' into themselves and so leaves hyperbolic distances invariant. Thus if $\alpha = \frac{1}{2}\pi$, and w_1, w_2 are on the positive axis, we have

$$d(w_1, w_2; D) < \tfrac{1}{2}\log K + \tfrac{1}{2}\log 2. \tag{3.9}$$

Next we make a transformation

$$\zeta = \log z = \xi + i\eta.$$

This maps D onto the rectangle P given by

$$-\log K < \xi < \log K, \quad |\eta| < \alpha.$$

Since $\log K > \frac{1}{2}\pi$, P contains the disk P_0: $|\zeta| < \frac{1}{2}\pi$. If w_1' is any point on $|z| = 1$, $|\arg z| < \frac{1}{2}\pi(1-\delta)$, then w_1' corresponds to

$$\zeta_1' = i\eta, \quad \text{where } |\eta| < \tfrac{1}{2}\pi(1-\delta),$$

and w_1 corresponds to $\zeta_1 = 0$. Thus

$$d(w_1, w_1'; D) \leqslant d(0, i\eta; P_0) = \tfrac{1}{2}\log\frac{1+2|\eta|/\pi}{1-2|\eta|/\pi} = \tfrac{1}{2}\log\frac{\pi+2|\eta|}{\pi-2|\eta|} < \tfrac{1}{2}\log\frac{2-\delta}{\delta}. \tag{3.10}$$

A similar argument shows that if w_2' is any point such that

$$|w_2'| = r, \quad |\arg w_2'| < \tfrac{1}{2}\pi(1-\delta), \quad \text{where } e^{\frac{1}{2}\pi}/K < r < Ke^{-\frac{1}{2}\pi},$$

then

$$d(w_2, w_2'; D) < \tfrac{1}{2}\log(2/\delta). \tag{3.11}$$

Thus in all cases we deduce, if $\alpha = \frac{1}{2}\pi$, using (3.9) to (3.11), that

$$\begin{aligned} d(w_1', w_2'; D) &\leqslant d(w_1', w_1; D) + d(w_1, w_2; D) + d(w_2, w_2'; D) \\ &< \tfrac{1}{2}\log K + \tfrac{1}{2}\log 2 + \log(2/\delta) \\ &< \tfrac{1}{2}\log K + \log(3/\delta), \end{aligned}$$

so that Lemma 5 is proved if $\alpha = \frac{1}{2}\pi$, $R = 1$.

In the general case we consider the transformation

$$Z = (z/R)^{\pi/(2\alpha)}.$$

This maps D, D' onto the domains we have just considered with the same value of δ, and $K^{\pi/(2\alpha)}$ instead of K. Since hyperbolic distances are invariant under conformal maps, we deduce Lemma 5 in the general case.

4. *Proof of Theorem* 1

We shall deduce Theorem 1 from the following somewhat stronger result.

THEOREM 3. *Suppose that $f(z)$ is regular in $S(\alpha, \beta)$, and that there exists an increasing sequence r_ν of positive numbers satisfying* (2.1), (2.3), *and*

$$r_{\nu+1} \leqslant r_\nu^{\eta}, \quad \text{where } \nu = 1, 2, \ldots, \tag{4.1}$$

and that $k(0, S') \geqslant \rho$. Suppose also that $0 < \rho < \pi/(\beta-\alpha)$, and that

$$1 < \eta < \frac{1+(\beta-\alpha)\rho/\pi}{1-(\beta-\alpha)\rho/\pi}. \tag{4.2}$$

Then we have for every a with at most one exception $k(a, S) \geqslant \rho'$, where ρ' is the positive root of the equation

$$\left(\frac{(\beta-\alpha)\rho'}{\pi}+1\right)\left(\frac{(\beta-\alpha)(\rho'-\rho)}{\pi}+1\right) - \left\{\frac{(\beta-\alpha)\rho}{\pi}+1\right\}\frac{1}{\eta} = 0. \tag{4.3}$$

COROLLARY 1. *If $\rho = \pi/(\beta-\alpha)$ and* (4.1) *holds for some finite η then $k(a, S) \geqslant \rho\{\sqrt{(\frac{1}{4}+2/\eta)} - \frac{1}{2}\}$ for every a with at most one exception.*

COROLLARY 2. *If* $\rho < \pi/(\beta-\alpha)$ *and* (4.1) *and* (4.2) *hold then* $k(a,S) > 0$ *for every* a *with at most one exception.*

COROLLARY 3. *If* $\rho < \pi/(\beta-\alpha)$ *and* (2.2) *holds then* $k(a,S) \geqslant \rho$ *for every* a *with at most one exception.*

Before proving Theorem 3 we shall deduce the corollaries from Theorem 3. We note that Corollary 3 is simply a restatement of Theorem 1.

Suppose first that $\rho < \pi/(\beta-\alpha)$. The condition (4.2) ensures that the left-hand side of (4.3) is negative when $\rho' = 0$ and positive when $\rho' = \rho$. Thus the equation (4.3) has a root ρ', such that $0 < \rho' < \rho$ and Corollary 2 follows from Theorem 3. Also if $\eta = 1$ we have $\rho' = \rho$ in (4.3) and so ρ' approaches ρ as η tends to 1. If (2.2) holds we may apply Theorem 3 with η arbitrarily close to 1 and obtain the conclusion with ρ' arbitrarily close to ρ and this yields Corollary 3, i.e. Theorem 1.

Finally, if $\rho = \pi/(\beta-\alpha)$ and $\eta > 1$, then the final term of (4.2) is no restriction, so with fixed η we may apply Theorem 3 with $\rho-\varepsilon$ instead of ρ, where ε is a sufficiently small positive number. Letting ε tend to zero we obtain the conclusion $k(a,S) \geqslant \rho'$ for every a with at most one exception, where ρ' is the positive root of the equation

$$\frac{\rho'}{\rho}\left(1+\frac{\rho'}{\rho}\right) = \frac{2}{\eta}$$

that is,

$$\rho' = \rho\{\sqrt{(\tfrac{1}{4}+2/\eta)}-\tfrac{1}{2}\}.$$

This is Corollary 1.

Letting η tend to 1 we obtain the conclusion of Theorem 1 also in this case. On the other hand, if $\rho > \pi/(\beta-\alpha)$, then Theorem 1 is a consequence of Theorem A. Thus assuming Theorem 3, we have proved Theorem 1 in all cases.

4.1. *Proof of Theorem* 3

To prove Theorem 3, we may suppose, without loss of generality, that $\alpha = -\beta$, since otherwise we may consider $f(ze^{-i\theta})$ instead of $f(z)$, where $\theta = \frac{1}{2}(\alpha+\beta)$. Next we suppose that Theorem 3 is false and obtain a contradiction. Suppose then that $k(a,S) < \rho_1$ and $k(b,S) < \rho_1$, where $a \neq b$ and $\rho_1 < \rho'$. We consider

$$F(z) = \frac{f(z)-a}{b-a}$$

instead of $f(z)$. For this function we have the conditions

$$k(0,S) < \rho_1, \quad k(1,S) < \rho_1, \quad k(a_0,S') \geqslant \rho,$$

where $a_0 = a/(a-b)$. Also if $\rho_2 < \rho$, we have, for all sufficiently large ν,

$$\log M(r_\nu, S') > r_\nu^{\rho_2}. \tag{4.4}$$

We write α instead of β so that D is the angle $S(-\alpha,\alpha)$ and S' the angle $S(-\alpha+\delta,\alpha-\delta)$ for some positive δ. We also have, for large r,

$$n(r,0,S)+n(r,1,S) < r^{\rho_1}, \tag{4.5}$$

and we choose ρ_2 so near ρ that $\rho_1 < \rho_2 < \rho$ and

$$\eta\left(\frac{2\alpha\rho_1}{\pi}+1\right)\left(\frac{2\alpha(\rho_1-\rho_2)}{\pi}+1\right) < \left(\frac{2\alpha\rho_2}{\pi}+1\right). \tag{4.6}$$

This is possible since $\rho_1 < \rho'$ and since (4.3) holds.

We choose numbers η_1 and η_2 such that

$$1 < \eta_1 < \frac{\rho_2+\pi/(2\alpha)}{\rho_1+\pi/(2\alpha)}, \quad 1 < \eta_2 < \frac{\pi/(2\alpha)}{\pi/(2\alpha)+\rho_1-\rho_2}, \tag{4.7}$$

and

$$\eta_1\eta_2 = \eta. \tag{4.8}$$

This is possible in view of (4.6). For every large v we shall apply Lemma 4 twice. We first take for D the domain

$$r_v/K < |z| < Kr_v, \quad |\arg z| < \alpha,$$

where

$$K = e^{\alpha}r_v^{\eta_1-1}, \tag{4.9}$$

and for z_1 a point on $|z_1| = r_v$, $|\arg z_1| < \alpha(1-\delta)$, such that

$$\log|F(z_1)| > r_v^{\rho_2}.$$

This is possible in view of (4.4). Let D' be the domain

$$e^{\alpha}r_v/K < |z| < r_vKe^{-\alpha}.$$

Then it follows from Lemma 5, that if z_2 is any point of D', we have

$$d(z_1, z_2; D) < \frac{\pi}{4\alpha}\log K + \log\frac{3}{\delta}.$$

Also by (4.5) the number of roots of the equations $F(z) = 0, 1$ in D total at most $(Kr_v)^{\rho_1}$. To show that the hypotheses and so the conclusions of Lemma 4 hold, we must prove that

$$r_v^{\rho_2} > A_1\{\log C + 1 + (Kr_v)^{\rho_1}\}\exp\left\{\frac{\pi}{2\alpha}\log K + 2\log\frac{3}{\delta}\right\},$$

with $C > |a_0|$. In view of (4.9) this is satisfied for large v if

$$r_v^{\rho_2} > 2A_1\left(\frac{3}{\delta}\right)^2(e^{\alpha}r_v^{\eta_1})^{\rho_1}(e^{\alpha}r_v^{\eta_1-1})^{\pi/(2\alpha)},$$

which is true since, by (4.7),

$$\eta_1\rho_1 + (\eta_1-1)\pi/(2\alpha) < \rho_2. \tag{4.10}$$

We deduce that the equation $F(z) = a_0$ has at most $(Kr_v)^{\rho_1}$ roots in

$$\frac{e^{\alpha}r_v}{K} < |z| < e^{-\alpha}Kr_v, \quad |\arg z| < \alpha(1-\delta).$$

In particular, for $r_v \leqslant r \leqslant r_v^{\eta_1}$, the equation has at most $(Kr_v)^{\rho_1} = (e^{\alpha}r_v^{\eta_1})^{\rho_1}$ roots in

$$|\arg z| < \alpha(1-\delta), \quad \tfrac{1}{2}r < |z| < r. \tag{4.11}$$

Thus it follows from (4.10) that for this range of r, the number of roots is $O(r^{\rho_2})$ in (4.11).

Next we shall obtain the same conclusion for $r_\nu^{\eta_1} \leqslant r \leqslant r_{\nu+1}$. For this purpose we apply Lemma 4 again, taking for D the domain

$$\frac{r_\nu^{\eta_1}}{2K} < |z| < \tfrac{1}{2}Kr_\nu^{\eta_1}, \quad |\arg z| < \alpha(1-\delta),$$

where

$$K = 2e^\alpha r_{\nu+1}/r_\nu^{\eta_1} < 2e^\alpha r_{\nu+1}^{(1-1/\eta_2)},$$

by (4.1) and (4.8). The number of roots in D of the equations $F(z) = 0, 1$ does not exceed $(e^\alpha r_{\nu+1})^{\rho_1}$ by (4.5). We take for z_1 a point on

$$|z| = r_{\nu+1}, \quad |\arg z| < \alpha(1-\delta),$$

such that $\log|f(z_1)| > r_{\nu+1}^{\rho_2}$. Using Lemma 5, we see that the domain D' of Lemma 4 includes the sectorial region

$$|\arg z| < \alpha(1-\delta), \quad \tfrac{1}{2}r_\nu^{\eta_1} < |z| < r_{\nu+1}, \tag{4.12}$$

provided that

$$r_{\nu+1}^{\rho_2} > 2A_1\{\log C + 1 + (e^\alpha r_{\nu+1})^{\rho_1}\}\exp\left\{\frac{\pi}{2\alpha}\log K + 2\log\frac{3}{\delta}\right\},$$

and this is the case for large ν, since by (4.7), $\rho_1 + (\pi/2\alpha)(1-1/\eta_2) < \rho_2$. We can therefore apply Lemma 4, and deduce that the equation $F(z) = a_0$ has at most $(e^\alpha r_{\nu+1})^{\rho_1}$ roots in (4.11) and hence $O(r_\nu^{\eta_1\rho_2})$ roots, since by (4.1) and (4.8) we have $r_{\nu+1}^{\rho_1} \leqslant (r_\nu^{\eta_1})^{\eta_2\rho_1}$ and, by (4.7),

$$\eta_2\rho_1 - \rho_2 < (\eta_2 - 1)(\rho_2 - \pi/(2\alpha)) < 0.$$

Thus, in particular, $F(z) = a_0$ has $O(r^{\rho_2})$ roots in (4.11) for $r_\nu^{\eta_1} \leqslant r \leqslant r_{\nu+1}$. We had previously obtained the same conclusion for $r_\nu \leqslant r \leqslant r_\nu^{\eta_1}$, and deduce that this result is true for all large r. By adding the roots of $F(z) = a_0$ over the sectors

$$2^{-k}r < |z| < 2^{1-k}r, \quad |\arg z| < \alpha(1-\delta)$$

for $k = 1$ to ∞, we deduce that $F(z) = a_0$ has $O(r^{\rho_2})$ roots in

$$|\arg z| < \alpha(1-\delta), \quad |z| < r,$$

so that $k(a_0, S') \leqslant \rho_2$, contrary to hypothesis. This contradiction proves Theorem 3 and so completes the proof of Theorem 1 also.

5. *Borel directions*

Suppose that $f(z)$ is meromorphic in an angle $S(\alpha, \beta)$ and that $\alpha < \theta_0 < \beta$. We say that the ray $\arg z = \theta_0$ is a *Borel direction of order ρ for f* [5, p. 32] if, for every small positive δ and every a in the closed plane with at most two exceptions, we have

$$k\{a, S(\theta_0 - \delta, \theta_0 + \delta)\} \geqslant \rho.$$

The following strengthened form of Theorem A is due to Valiron [5].

THEOREM B. *If $f(z)$ satisfies the hypotheses of Theorem A, then $f(z)$ has a Borel direction of order $\rho = k(S')$ in S'.*

Valiron's technique yields an analogous extension of Theorem 1. This is

THEOREM 4. *If $f(z)$ satisfies the hypotheses of Theorem 1, then $f(z)$ has a Borel direction of order ρ in S'.*

We need a preliminary result due to Valiron [5, p. 31]. In our terminology a slightly weaker form of this result is

LEMMA 6. *Suppose that $f(z)$ is meromorphic in an angle S and that $k(a, S) \leqslant \sigma$, for three distinct values $a = a_j$ in the closed plane, where $\sigma > 0$. Then*

$$k(a, S') \leqslant \sigma$$

for every S' and all complex a outside a set E of measure zero.

Theorem 2 shows that the exceptional set E need not be empty. We can now prove Theorem 4. Suppose that, contrary to this, no ray $\arg z = \theta$ for $\alpha' \leqslant \theta \leqslant \beta'$ is a Borel direction of order ρ for f, where f satisfies the hypotheses of Theorem 1. Since f is regular this implies that for each such θ there exists a positive δ and two finite complex values a_1 and a_2 such that

$$k\{a_j, S(\theta-\delta, \theta+\delta)\} \leqslant \rho(\theta) < \rho, \quad \text{for } j = 1, 2.$$

Using Lemma 6, we deduce that $k\{a, S(\theta-\frac{1}{2}\delta, \theta+\frac{1}{2}\delta)\} \leqslant \rho(\theta)$ for a outside $E(\theta)$, where $E(\theta)$ has measure zero.

It follows from the Heine–Borel theorem that there are a finite number of θ's, θ_1 to θ_N in $\alpha' \leqslant \theta \leqslant \beta'$, such that the corresponding intervals $(\theta_j-\frac{1}{2}\delta_j, \theta_j+\frac{1}{2}\delta_j)$ cover (α', β'). If E_j are the corresponding exceptional sets $E(\theta_j)$ and $E = \bigcup_{j=1}^{N} E(\theta_j)$, then we deduce that for a outside E we have

$$k(a, S_1) \leqslant \rho' = \max_{j=1,\ldots,N} \rho(\theta_j) < \rho.$$

Here E has measure zero and

$$S_1 = \bigcup_{j=1}^{N} S(\theta_j-\tfrac{1}{2}\delta_j,\ \theta_j+\tfrac{1}{2}\delta_j),$$

so that S' lies in the interior of S_1. But this contradicts Theorem 1, with S_1 instead of S, since Theorem 1 asserts that $k(a, S_1) \geqslant \rho$ for every a with at most one exception. This contradiction proves Theorem 4.

6. *Construction of the counter-example*

We now start the proof of Theorem 2. The rough idea is as follows. We express $f(z)$ as a product of two functions $f(z) = f_1(z)f_2(z)$. The function $f_1(z)$ behaves rather like a Blaschke product. It is uniformly bounded in the right half-plane and is small and has zeros of order ρ in a sequence (class I) of annuli; in these annuli $f_2(z)$ is not too large so that $f(z)$ is small. On the other hand, in another sequence (class II) of annuli $f_2(z)$ behaves like a classical Lindelöf function of order ρ, with negative zeros, while $f_1(z)$ is not too small. Thus $f_2(z)$ and hence $f(z)$ is large of order ρ in the class II annuli. This shows that $k(0, S) \geqslant \rho$ and $k(S) \geqslant \rho$. For $a \neq 0$, the equation $f(z) = a$ can have at most a finite number of roots outside an intermediate sequence (class III) of annuli, where $f(z)$ is neither large nor small and in these annuli both $f_1(z)$ and $f_2(z)$ vary slowly so that $k(a, S') \leqslant \rho' < \rho$. In this section we define $f_1(z), f_2(z)$, and $f(z) = f_1(z)f_2(z)$. The relevant inequalities for these three functions are established in §§ 7 to 9 and the proof of Theorem 2 is completed in § 10.

We suppose that (2.4) and (2.5) hold and write

$$\eta = K^2. \tag{6.1}$$

By choosing a subsequence if necessary we may assume that $r_1 > 4$,

$$r_\nu > 2r_{\nu-1}{}^{\{2K-(1+\rho)\}/(1-\rho)}, \quad \text{for } \nu \geqslant 2,$$

and

$$r_\nu > 2r_{\nu-1}{}^{K(1+\rho)/[2-K(1-\rho)]}, \quad \text{for } \nu \geqslant 2.$$

Using these inequalities in turn we obtain

$$r_{\nu-1}{}^{2\eta} < r_{\nu-1}{}^{K(1+\rho)}(\tfrac{1}{2}r_\nu)^{K(1-\rho)} < (\tfrac{1}{2}r_\nu)^2. \tag{6.2}$$

We now set

$$s_\nu = r_\nu{}^K \tag{6.3}$$

and

$$t_\nu = r_\nu{}^{\rho+\eta(1-\rho)}\log r_\nu. \tag{6.4}$$

It follows from (6.1), (2.4), and (2.5) that

$$\rho+\eta(1-\rho)-K = \rho+K^2(1-\rho)-K = (K-1)(K-(K+1)\rho) < 0,$$

so that

$$t_\nu = o(s_\nu) \quad \text{as } \nu \to \infty. \tag{6.5}$$

We now define

$$f_1(z) = (e^{-2\pi z}-1)\prod\nolimits_1\left\{\frac{ze^{i\delta}-in}{z-in}\right\}, \tag{6.6}$$

where the product $\prod_1$ is taken over all integers n which satisfy

$$s_\nu \leqslant n \leqslant s_\nu+(s_\nu)^\rho \quad \text{for some } \nu. \tag{6.7}$$

Next we write

$$f_2(z) = \prod\nolimits_2(1+zn^{-1/\rho}), \tag{6.8}$$

where the product $\prod_2$ is taken over all integers n which satisfy

$$r_\nu{}^\eta \leqslant n^{1/\rho} \leqslant r_{\nu+1} \quad \text{for some } \nu, \tag{6.9}$$

and set

$$f(z) = f_1(z)f_2(z). \tag{6.10}$$

We shall write C for positive constants depending on δ, η, ρ, and C_1, C_2 for particular such constants. We set $z = re^{i\theta}$.

7. *Estimates for* $f_1(z)$

We proceed to prove

LEMMA 7. *If ν is large and $|\theta| \leqslant \frac{1}{2}\pi - \frac{2}{3}\delta$, then*

$$\log|f_1(z)| < -C_1\frac{r}{S_\nu{}^{(1-\rho)}} \quad \textit{for } r_\nu \leqslant r \leqslant s_\nu, \tag{7.1}$$

and

$$\log|f_1(z)| < -C_1\frac{s_\nu^{(1+\rho)}}{r}, \quad \textit{for } s_\nu \leqslant r \leqslant r_\nu^{\eta}. \tag{7.2}$$

If $|\theta| \leqslant \frac{1}{2}\pi - \frac{1}{2}\delta$, *then*

$$\log|f_1(z)| > -C_2\frac{r}{s_\nu^{(1-\rho)}}, \quad \textit{for } r_\nu \leqslant r \leqslant \tfrac{1}{2}s_\nu, \tag{7.3}$$

$$\log|f_1(z)| > -C_2\frac{s_\nu^{(1+\rho)}}{r}, \quad \textit{for } 4s_\nu \leqslant r \leqslant r_\nu^{\eta}, \tag{7.4}$$

and, for some branch of the logarithm,

$$|\log|f_1(z)|| < C_2 r^{\rho-(K-1)(1-\rho)}, \quad \textit{for } r_\nu^{\eta} \leqslant r \leqslant r_{\nu+1}. \tag{7.5}$$

Finally, $f_1(z)$ *is entire and*

$$|f_1(z)| < 2, \quad \textit{for } |\theta| < \tfrac{1}{2}(\pi-\delta) \textit{ and } 0 < r < \infty. \tag{7.6}$$

We note that for $|\theta| \leqslant \frac{1}{2}\pi - \frac{1}{2}\delta$, we have $|e^{-2\pi z}| < 1$ and

$$\left|\frac{z-ine^{-i\delta}}{z-in}\right| < 1, \quad \text{for every integer } n.$$

Thus $|f_1(z)| < 1 + |e^{-2\pi z}| < 2$ in this angle. This proves (7.6). Also

$$\begin{aligned} 1-\left|\frac{z-ine^{-i\delta}}{z-in}\right|^2 &= \frac{|z-in|^2-|z-ine^{-i\delta}|^2}{|z-in|^2} \\ &= \frac{2rn(\sin(\theta+\delta)-\sin\theta)}{|z-in|^2} \\ &= \frac{4rn\cos(\theta+\frac{1}{2}\delta)\sin\frac{1}{2}\delta}{|z-in|^2}. \end{aligned}$$

Thus if $|\theta| \leqslant \frac{1}{2}\pi - \frac{2}{3}\delta$, we deduce that $\cos(\theta+\frac{1}{2}\delta) > \sin(\delta/6)$, and

$$1-\frac{4rn}{|z-in|^2} < \left|\frac{z-ine^{-i\delta}}{z-in}\right|^2 < 1-\frac{4rn\sin^2(\delta/6)}{|z-in|^2}. \tag{7.7}$$

Suppose first that $r \leqslant s_\nu$ and that n lies in the range (6.7) for this value of ν. Then $Cs_\nu < |z-in| < Cs_\nu$. Thus if $\prod_1(z)$ denotes the product in (6.6), (7.7) yields

$$\log|\textstyle\prod_1(z)| < -CNr/s_\nu,$$

where N is the number of integers in the range (6.7). Since $s_\nu^{\rho} > 4^{\frac{1}{2}} = 2$, we deduce that $N \geqslant \frac{1}{2}s_\nu^{\rho}$, so that

$$\log|\textstyle\prod_1(z)| < -Crs_\nu^{\rho-1}.$$

If $r \geqslant r_\nu$, we see that $rs_\nu^{\rho-1} \geqslant r_\nu^{1-K(1-\rho)} > r_\nu^{1/3}$, in view of (2.4), (2.5), and (6.1). Using (7.6) with (6.6) we deduce (7.1). We also see that, for $r \geqslant s_\nu$,

$$Cr < |z-in| < Cr,$$

so that in this case (6.7) and (7.7) imply that

$$\log|\textstyle\prod_1(z)| < -CNs_\nu/r < -C(s_\nu)^{1+\rho}/r,$$

and this yields (7.2) in view of (7.6) and $K(1+\rho) > \eta$.

Next we turn to the lower bounds for $|f_1(z)|$, which are a little trickier. We suppose first that

$$(7.8) \qquad r_\mu \leqslant r \leqslant r_{\mu+1}$$

and estimate the contribution to the product in (6.6) from the ranges (6.7) for various ν. Suppose first that $\nu > \mu$. Then if n lies in the range (6.7) we have, recalling (6.3),

$$n \geqslant s_\nu = r_\nu{}^K \geqslant r_{\mu+1}{}^K \geqslant r^K.$$

Thus if μ is large, we have for all θ, since $K > 1$,

$$\left|\log\left(\frac{ze^{i\delta}-in}{z-in}\right)\right| < \frac{Cr}{n} \leqslant \frac{Cr}{s_\nu}.$$

Since $\rho < 1$, and $s_{\nu+1} > 2s_\nu$ by (6.2) and (6.3), the product in (6.6) converges absolutely and locally uniformly. In fact the contribution to $|\log\prod_1(z)|$ from the range (6.7) is at most $Crs_\nu{}^\rho/s_\nu = Crs_\nu{}^{\rho-1}$. Thus

$$\textstyle\sum_1 \left|\log\left(\dfrac{ze^{i\delta}-in}{z-in}\right)\right| < Crs_{\mu+1}{}^{\rho-1} = Crr_{\mu+1}{}^{K(\rho-1)},$$

where the sum in $\sum_1$ is taken for n in all the ranges ν with $\nu > \mu$.

Next if $\nu < \mu$, we see similarly that the contribution to $\log|1/\prod_1(z)|$ from n in the range (6.7) is at most

$$C\textstyle\sum n/r < Cs_\nu{}^{1+\rho}/r.$$

Thus the contribution from all the ranges (6.7) with $\nu < \mu$ is at most

$$Cs_{\mu-1}{}^{1+\rho}/r = Cr_{\mu-1}{}^{K(1+\rho)}/r.$$

Finally, if r lies in the range (7.8), $r < \frac{1}{2}s_\mu$, and n lies in the range (6.7) with $\nu = \mu$, we have

$$|z-in| > \tfrac{1}{2}n > \tfrac{1}{2}s_\mu.$$

Thus we see that the contribution to $\log|1/\prod_1(z)|$ from the n in the range (6.7) with $\nu = \mu$, is at most $Crs_\mu{}^{\rho-1} = Crr_\mu{}^{K(\rho-1)}$. Also $|e^{-2\pi z}| < C < 1$, if $|\theta| < \frac{1}{2}\pi - \frac{1}{2}\delta$ and $r > 1$. Thus we obtain for $r_\mu \leqslant r \leqslant \frac{1}{2}s_\mu$ and $|\theta| < \frac{1}{2}\pi - \frac{1}{2}\delta$,

$$\begin{aligned}\log|f_1(z)| &> -C(1+rr_\mu{}^{K(\rho-1)}+rr_{\mu+1}{}^{K(\rho-1)}+r_{\mu-1}{}^{K(1+\rho)}/r)\\ &> -C(1+r_{\mu-1}{}^{K(1+\rho)}/r+rr_\mu{}^{K(\rho-1)}).\end{aligned}$$

In view of (6.2) and (7.8), we have

$$r_{\mu-1}{}^{K(1+\rho)}/r \leqslant r_{\mu-1}{}^{K(1+\rho)}/r_\mu < r_\mu r_\mu{}^{K(\rho-1)} \leqslant rr_\mu{}^{K(\rho-1)}.$$

Thus

$$\log|f_1(z)| > -C(1+rr_\mu{}^{K(\rho-1)}).$$

Since $K(1-\rho) < 1$, we deduce (7.3).

Suppose next that r lies in the range (7.8) and $4s_\mu \leqslant r \leqslant r_\mu{}^\eta$. Then if n lies in the range (6.7) with $\nu = \mu$, we have $n < 2s_\mu$, so that

$$\left|\log\left(\frac{1-ine^{-i\delta}/z}{1-in/z}\right)\right| < Cn/r.$$

Thus the contribution of this range of n to $|\log\prod_1(z)|$ is at most $Cs_\mu{}^{(1+\rho)}/r = Cr_\mu{}^{K(1+\rho)}/r$ in this case. We recall our earlier work to get (7.3) and obtain

$$\begin{aligned}\log|f_1(z)| &> -C(1+r_{\mu-1}{}^{K(1+\rho)}/r+r_\mu{}^{K(1+\rho)}/r + rr_{\mu+1}{}^{K(\rho-1)}) \\ &> -C(1+r_\mu{}^{K(1+\rho)}/r),\end{aligned}$$

in view of the first inequality in (6.2) with $\nu = \mu+1$, and $r \leqslant r_\mu{}^\eta$. This proves (7.4), since $K(1+\rho) > \eta = K^2$.

Suppose finally that $r_\mu{}^\eta \leqslant r \leqslant r_{\mu+1}$. In this case our previous estimates yield

$$\log|f_1(z)| > -C(1+r_\mu{}^{K(1+\rho)}/r+rr_{\mu+1}{}^{K(\rho-1)}). \tag{7.9}$$

We note that $\eta = K^2$, so that

$$r_\mu{}^{K(1+\rho)}/r \leqslant r_\mu{}^{K(1+\rho-K)}.$$

Also $0 < 1+\rho-K < \rho$ in view of (2.4) and (2.5). Thus

$$r_\mu{}^{K(1+\rho-K)} \leqslant r^{(1+\rho-K)/K} < r^{\rho-(K-1)} < r^{\rho-(K-1)(1-\rho)}.$$

Hence $r_\mu{}^{K(1+\rho)}/r < r^{\rho-(K-1)(1-\rho)}$. Next since $r \leqslant r_{\mu+1}$, we have

$$rr_{\mu+1}{}^{K(\rho-1)} < r^{1+K(\rho-1)} = r^{\rho-(K-1)(1-\rho)}.$$

Thus (7.9) and (7.6) yield (7.5).

We have seen that the product $\prod_1$ is absolutely convergent. Also $f_1(z)$ is regular in the plane except possibly at the points $z = in$, and the poles at these points in the product $\prod_1(z)$ are cancelled out by the zeros of $e^{-2\pi z}-1$. Thus $f_1(z)$ is an entire function. Further, $f_1(z)$ has order 1, mean type. For this is true of the factor $(e^{-2\pi z}-1)$ and the product $\prod_1$ has order ρ less than 1. Thus the proof of Lemma 7 is complete.

8. *The estimates for $f_2(z)$*

We next prove

LEMMA 8. *We have for $z = re^{i\theta}$, with $|\theta| < \frac{1}{2}\pi-\frac{1}{2}\delta$, and large ν,*

$$C_3r^\rho < \log|f_2(z)| < C_4r^\rho, \quad \text{if } r_\nu{}^\eta \leqslant r \leqslant r_{\nu+1}, \tag{8.1}$$

$$C_5r^\rho(1+\log(r/r_\nu)) < \log|f_2(z)| < C_6r_\nu{}^\rho(1+\log(r/r_\nu)), \text{ if } r_\nu \leqslant r \leqslant t_\nu, \tag{8.2}$$

$$C_7rr_\nu{}^{\eta(\rho-1)} < \log|f_2(z)| < C_8rr_\nu{}^{\eta(\rho-1)}, \quad \text{if } t_\nu \leqslant r \leqslant r_\nu{}^\eta. \tag{8.3}$$

We suppose that t_μ is defined by (6.4) and that

$$t_\mu \leqslant r \leqslant t_{\mu+1}. \tag{8.4}$$

First we derive the lower bounds in (8.1) to (8.3).

In considering the ranges (6.9) for different values of ν, we note that since $|\theta| < \frac{1}{2}\pi$, we have

$$|1+zn^{-1/\rho}| > 1. \tag{8.5}$$

Thus when obtaining lower bounds for $|f_2(z)|$ we may confine ourselves to the single range $\nu = \mu$ in (6.9). Suppose first that

$$t_\mu \leqslant r \leqslant r_\mu{}^\eta. \tag{8.6}$$

We note that, for $|\theta| < \frac{1}{2}(\pi - \delta)$,

$$|1+zn^{-1/\rho}|^2 = 1 + \frac{2r\cos\theta}{n^{1/\rho}} + \frac{r^2}{n^{2/\rho}} \geqslant \left(1 + \frac{r\sin\frac{1}{2}\delta}{n^{1/\rho}}\right)^2. \tag{8.7}$$

Thus

$$\log|1+zn^{-1/\rho}| \geqslant \log\left(1 + \frac{r\sin(\frac{1}{2}\delta)}{n^{1/\rho}}\right) \geqslant Crn^{-1/\rho}.$$

Since by (6.2) we have $r_{\mu+1} > 2r_\mu{}^\eta$, we deduce from (8.5) that

$$\log|f_2(z)| \geqslant Cr\sum n^{-1/\rho} > CNrr_\mu{}^{-\eta},$$

where N is the number of integers n satisfying $r_\mu{}^\eta \leqslant n^{1/\rho} \leqslant 2r_\mu{}^\eta$, that is $r_\mu{}^{\rho\eta} \leqslant n \leqslant 2^\rho r_\mu{}^{\rho\eta}$. Thus, when μ is large,

$$N \geqslant (2^\rho - 1)r_\mu{}^{\rho\eta} - 1 > Cr_\mu{}^{\rho\eta},$$

and

$$\log|f_2(z)| > Crr_\mu{}^{(\rho-1)\eta}$$

in the range (8.6), and this proves the lower bound in (8.3).

Next suppose that

$$r_\mu{}^\eta \leqslant r \leqslant r_{\mu+1}. \tag{8.8}$$

We now confine ourselves to estimating the product for $f_2(z)$ over those integers n which lie in the range (6.9) with $\nu = \mu$ and also satisfy $2^{-\frac{1}{2}}r < n^{1/\rho} < 2^{\frac{1}{2}}r$. By considering separately the cases $r \leqslant 2^{\frac{1}{2}}r_\mu{}^\eta \leqslant 2^{-\frac{1}{2}}r_{\mu+1}$ and $r > 2^{\frac{1}{2}}r_\mu{}^\eta$, we see that the number N of these integers n satisfies

$$N \geqslant r^\rho(1 - 2^{-\rho/2}) - 1 > Cr^\rho.$$

Also, for n in this range, $\log|1+zn^{-1/\rho}| > C$. Thus in the range (8.8) we have, using (8.5),

$$\log|f_2(z)| > CN > Cr^\rho,$$

which yields the left-hand side of (8.1).

Finally, suppose that

$$r_{\mu+1} \leqslant r \leqslant t_{\mu+1}. \tag{8.9}$$

We confine ourselves now to the product over integers n which satisfy

$$\tfrac{1}{2}r_{\mu+1} \leqslant n^{1/\rho} \leqslant r_{\mu+1}. \tag{8.10}$$

These lie in the range (6.9) with $\nu = \mu$, and (8.7) yields

$$|1+zn^{-1/\rho}| \geqslant 1 + rn^{-1/\rho}\sin\tfrac{1}{2}\delta = 1 + Crn^{-1/\rho} \geqslant 1 + C_0 r/r_{\mu+1}.$$

Thus if N is the number of integers in the range (8.10) we obtain

$$\log|f_2(z)| \geqslant N\log(1 + C_0 r/r_{\mu+1}) \geqslant Cr_{\mu+1}{}^\rho \log(1 + C_0 r/r_{\mu+1}).$$

If $r \leqslant C_0{}^{-2}r_{\mu+1}$, we obtain $\log|f_2(z)| \geqslant Cr_{\mu+1}{}^\rho$, which yields the lower bound in (8.2).

If $C_0^{-2}r_{\mu+1} \leqslant r \leqslant t_{\mu+1}$ then $C_0 r/r_{\mu+1} \geqslant (r/r_{\mu+1})^{\frac{1}{2}}$ and we obtain

$$\log|f_2(z)| \geqslant C r_{\mu+1}{}^{\rho}\tfrac{1}{2}\log(r/r_{\mu+1}),$$

which again yields the lower bound in (8.2). This completes the proof of the lower bounds in Lemma 8.

To establish the upper bounds we suppose that $r_\mu \leqslant r \leqslant r_{\mu+1}$ and consider first the range $r_\mu{}^\eta \leqslant r \leqslant r_{\mu+1}$. In this range

$$\log|f_2(z)| \leqslant \sum_{n=1}^{\infty} \log(1+rn^{-1/\rho}) \leqslant \log(1+r) + \int_1^\infty \log(1+rt^{-1/\rho})\,dt.$$

We set $x = rt^{-1/\rho}$ in the integral and obtain

$$\log|f_2(z)| \leqslant \log(1+r) + \rho r^\rho \int_0^\infty \frac{\log(1+x)}{x^{\rho+1}}\,dx = Cr^\rho + \log(1+r),$$

and this yields the right-hand side of (8.1).

Next suppose that

$$r_\mu \leqslant r \leqslant r_\mu{}^\eta. \tag{8.11}$$

Then recalling (6.9) we see that

$$\log|f_2(z)| \leqslant \textstyle\sum_1 \log(1+rn^{-1/\rho}) + \sum_2 \log(1+rn^{-1/\rho}),$$

where the sums $\sum_1, \sum_2$ are taken over the ranges $n \leqslant r_\mu{}^\rho$, and $n \geqslant r_\mu{}^{\eta\rho}$ respectively. Thus, in view of (6.9),

$$\textstyle\sum_2 \leqslant \sum_2 rn^{-1/\rho} = r\sum_2 n^{-1/\rho} \leqslant Cr(r_\mu{}^{\eta\rho})^{1-1/\rho} = Crr_\mu{}^{\eta\rho-\eta}. \tag{8.12}$$

Also if N is the integral part of $r_\mu{}^\rho$, then

$$\begin{aligned}\textstyle\sum_1 &\leqslant \log(1+r) + \int_1^{N+1} \log(1+rt^{-1/\rho})\,dt \\ &= \log(1+r) + \rho r^\rho \int_{r/(N+1)^{1/\rho}}^{r} x^{-\rho-1}\log(1+x)\,dx \\ &\leqslant \log(1+r) + Cr^\rho\left(\frac{(N+1)^{1/\rho}}{r}\right)^\rho \log\left(\frac{er}{(N+1)^{1/\rho}}\right) \\ &\leqslant \log(1+r) + C(N+1)\log\left(\frac{er}{(N+1)^{1/\rho}}\right).\end{aligned}$$

Thus

$$\textstyle\sum_1 \leqslant C\{\log(1+r) + r_\mu{}^\rho(\log(r/r_\mu)+1)\} < Cr_\mu{}^\rho(\log(r/r_\mu)+1). \tag{8.13}$$

Thus in the range (8.11) we obtain, from (8.12) and (8.13),

$$\log|f_2(z)| < C\{r_\mu{}^\rho \log(er/r_\mu) + rr_\mu{}^{\eta(\rho-1)}\}. \tag{8.14}$$

Suppose first that $r_\mu \leqslant r \leqslant t_\mu/\log r_\mu = r_\mu{}^{\rho+\eta(1-\rho)}$. Then $rr_\mu{}^{\eta(\rho-1)} \leqslant r_\mu{}^\rho$, and (8.14) yields

$$\log|f_2(z)| < Cr_\mu{}^\rho(\log(r/r_\mu)+2),$$

which implies the right-hand side of (8.2). Suppose next that

$$t_\mu/\log r_\mu \leqslant r \leqslant t_\mu.$$

Then (6.4) shows that $r/r_\mu \geqslant r_\mu^{(\eta-1)(1-\rho)}$. Thus, in view of (6.4) and since $\eta > 1$,

$$rr_\mu^{\eta(\rho-1)} \leqslant r_\mu^\rho \log r_\mu \leqslant Cr_\mu^\rho \log(r/r_\mu),$$

and (8.14) again yields the right-hand side of (8.2).

Suppose, finally that $t_\mu \leqslant r \leqslant r_\mu^\eta$. Then

$$\begin{aligned} r_\mu^\rho \log(er/r_\mu) &\leqslant r_\mu^\rho \log(r/r_\mu^{\rho+\eta(1-\rho)}) + Cr_\mu^\rho \log r_\mu \\ &\leqslant r_\mu^\rho(r/r_\mu^{\rho+\eta(1-\rho)}) + Cr_\mu^\rho \log r_\mu \\ &\leqslant C(rr_\mu^{\eta(\rho-1)} + t_\mu r_\mu^{\eta(\rho-1)}) \\ &\leqslant Crr_\mu^{\eta(\rho-1)}. \end{aligned}$$

Thus in this range (8.14) yields the right-hand side of (8.3) and the proof of Lemma 8 is complete.

9. *Estimates for $f(z)$*

We shall use the estimates of Lemmas 7 and 8 to show that $f(z)$ is either large or small except when $|z|$ is comparable to certain numbers p_ν and q_ν. Thus, except in this case, the equation $f(z) = a$ can have no solutions. Near $|z| = p_\nu$ or q_ν the order of f is at most ρ' (defined in (2.8)) and so is the order of the number of solutions of $f(z) = a$. We now make these estimates more precise.

We set

(9.1) $$p_\nu = r_\nu^{\rho+K(1-\rho)} \log r_\nu,$$

(9.2) $$q_\nu = r_\nu^{\frac{1}{2}\eta(1-\rho)+\frac{1}{2}K(1+\rho)}.$$

We note, in view of (2.4), (2.5), and (6.1), that

$$1 < \rho + K(1-\rho) < \rho + \eta(1-\rho),$$

so that, in view of (6.4),

(9.3) $$r_\nu = o(p_\nu) \quad \text{and} \quad p_\nu = o(t_\nu).$$

Similarly,

$$K = \tfrac{1}{2}K(1-\rho) + \tfrac{1}{2}K(1+\rho) < \tfrac{1}{2}\eta(1-\rho) + \tfrac{1}{2}K(1+\rho) < \eta.$$

Thus, in view of (6.3),

$$s_\nu = o(q_\nu) \quad \text{and} \quad q_\nu = o(r_\nu^\eta).$$

Using this and (6.2), (6.5), (9.3) we see that the following are in order of increasing magnitude:

(9.4) $$r_\nu,\ p_\nu,\ t_\nu,\ s_\nu,\ q_\nu,\ r_\nu^\eta,\ r_{\nu+1}.$$

We shall suppose we are given a number ε, such that $0 < \varepsilon < 1$.

LEMMA 9. *We have for $z = re^{i\theta}$, with $|\theta| < \frac{1}{2}\pi - \frac{2}{3}\delta$, and $\nu > \nu_0$,*

(9.5) $$|f(z)| < \varepsilon \quad \textit{if } C_9 p_\nu \leqslant r \leqslant C_{10} q_\nu,$$

and

(9.6) $$|f(z)| > 1/\varepsilon \quad \textit{if } C_{11} q_\nu \leqslant r \leqslant C_{12} p_{\nu+1}.$$

We consider first the range $r_\nu^\eta \leqslant r \leqslant r_{\nu+1}$. In this range we deduce from (8.1) and (7.5) that

$$\log|f(z)| = \log|f_1(z)| + \log|f_2(z)| > C_3 r^\rho - C_2 r^{\rho-(K-1)(1-\rho)},$$

and this proves (9.6) in this case.

Next recall (9.1) and suppose that $r_\nu \leqslant r \leqslant p_\nu$. Then (7.3) and (8.2) yield, in view of (9.4),

$$\log|f(z)| > C_5 r_\nu^\rho(1+\log(r/r_\nu)) - C_2 r r_\nu^{K(\rho-1)}$$
$$> \tfrac{1}{2}C_5 r_\nu^\rho(1+\log(r/r_\nu)) \tag{9.7}$$

if $C_2 r r_\nu^{K(\rho-1)} < \frac{1}{2}C_5 r_\nu^\rho(1+\log(r/r_\nu))$, that is, if

$$r < \frac{C_5}{2C_2} r_\nu^{\rho+K(1-\rho)}(1+\log(r/r_\nu)).$$

The condition is certainly satisfied if

$$r \leqslant \frac{C_5}{2C_2} r_\nu^{\rho+K(1-\rho)}.$$

If, on the other hand, $(C_5/2C_2)r_\nu^{\rho+K(1-\rho)} < r$, we have

$$\log\frac{r}{r_\nu} > (K-1)(1-\rho)\log r_\nu + O(1),$$

and so it is enough to assume that

$$r < \frac{C_5(K-1)(1-\rho)}{4C_2} r_\nu^{\rho+K(1-\rho)}\log r_\nu = C_{12} p_\nu,$$

in view of (9.1). Since the right-hand side of (9.7) is large for large ν, we deduce (9.6) in this case.

Suppose next that $p_\nu \leqslant r \leqslant t_\nu$. Then (7.1) and (8.2) yield, in view of (9.1), (9.4), and (6.3),

$$\log|f(z)| < -C_1 r r_\nu^{K(\rho-1)} + C_6 r_\nu^\rho(1+\log(r/r_\nu))$$
$$< -C_1 r r_\nu^{K(\rho-1)} + 2C_6(K-1) r_\nu^\rho \log r_\nu$$
$$< -\tfrac{1}{2}C_1 r r_\nu^{K(\rho-1)},$$

if $2C_6(K-1)r_\nu^\rho \log r_\nu < \frac{1}{2}C_1 r r_\nu^{K(\rho-1)}$, that is if

$$r > \frac{4C_6(K-1)}{C_1} r_\nu^{\rho+K(1-\rho)}\log r_\nu = \frac{4C_6(K-1)}{C_1} p_\nu = C_9 p_\nu.$$

Thus (9.5) holds in this case.

Suppose next that $t_\nu < r \leqslant s_\nu$. Then (7.1) and (8.3) yield for large ν, in view of (9.4) and (6.4),

$$\log|f(z)| < -C_1 r r_\nu^{K(\rho-1)} + C_8 r r_\nu^{\eta(\rho-1)}$$
$$< -\tfrac{1}{2}C_1 r r_\nu^{K(\rho-1)} < -r_\nu^{\rho+(\eta-K)(1-\rho)},$$

since $r > t_\nu$, and this yields (9.5).

Suppose now that $s_\nu < r \leqslant q_\nu$. Then (7.2) and (8.3) yield, in view of (9.4),

$$\log|f(z)| < -C_1\frac{r_\nu^{K(1+\rho)}}{r} + C_8 r r_\nu^{\eta(\rho-1)} < -\tfrac{1}{2}C_1\frac{r_\nu^{K(1+\rho)}}{r}$$

for large v, provided that

$$C_8 r r_v^{\eta(\rho-1)} < \frac{1}{2}\frac{C_1 r_v^{K(1+\rho)}}{r}.$$

By (9.2) this is equivalent to $r^2 < (C_1/(2C_8))q_v^2$, and this yields (9.5). Finally, if $q_v < r \leqslant r_v^\eta$, we have from (7.4), (8.3), (9.4), and (6.3),

$$\log|f(z)| > C_7 r r_v^{\eta(\rho-1)} - C_2 \frac{r_v^{K(1+\rho)}}{r} > \tfrac{1}{2}C_7 r r_v^{\eta(\rho-1)}$$

if $C_2 r_v^{K(1+\rho)}/r < \frac{1}{2}C_7 r r_v^{\eta(\rho-1)}$, that is if

$$r^2 > (2C_2/C_7)q_v^2,$$

and this proves (9.6) in this case and completes the proof of Lemma 9.

10. *Proof of Theorem* 2

It is evident from the construction of $f(z)$ that $f(z)$ is an entire function with zeros at $z = ine^{-i\delta}$, for n in the ranges (6.7). If we take $r = 2s_v$, then $f(z)$ has about $(\frac{1}{2}r)^\rho$ zeros in $|z| < r$, $|\arg z| \leqslant \frac{1}{2}\pi - \delta$. Thus if S, S' are the sectors

$$|\arg z| \leqslant \tfrac{1}{2}\pi - \tfrac{3}{4}\delta, \quad |\arg z| \leqslant \tfrac{1}{2}\pi - \delta,$$

then $k(0, S) = k(0, S') = \rho$.

Next it follows from Lemmas 7 and 8 and, in particular, from (7.5) and (8.1) that, for $z = re^{i\theta}$ in S and $r_v^\eta \leqslant r \leqslant r_{v+1}$, we have

$$\log|f(z)| > \tfrac{1}{2}C_3 r^\rho,$$

while $\log|f(z)| < Cr^\rho$ for other values of r. Thus $k(S) = k(S') = \rho$, and (2.6) holds.

We proved in Lemma 7 that $f_1(z)$ is an entire function. The function $f_2(z)$ has order ρ and so does the product $\prod_1$ in (6.6). Since $(e^{-2\pi z} - 1)$ has order 1, mean type, so does $f(z)$.

It remains to investigate the number of roots of the equation $f(z) = a$, when $a \neq 0$, and z lies in S.

It follows from Lemma 9, that if r is large, depending on a, then the only solutions of $f(z) = a$ in S lie in the annuli

$$C_{10} q_v < r < C_{11} q_v \quad \text{and} \quad C_{12} p_v < r < C_9 p_v.$$

It is convenient to expand these annuli slightly and to consider the sectors

$$\tfrac{1}{2}C_{10} q_v < r < 2C_{11} q_v, \quad |\theta| < \tfrac{1}{2}\pi - \tfrac{2}{3}\delta, \tag{10.1}$$

$$\tfrac{1}{2}C_{12} p_v < r < 2C_9 p_v, \quad |\theta| < \tfrac{1}{2}\pi - \tfrac{2}{3}\delta. \tag{10.2}$$

We need a final lemma.

LEMMA 10. *As $z \to \infty$ through the sectors* (10.1) *and* (10.2) *we have*

$$||\log|f(z)|| = o(r^{\rho'}), \tag{10.3}$$

where ρ' is given by (2.8).

Suppose first that (10.1) holds. Then (9.4), (7.2), (7.4), (6.3), (9.2), and (8.3) show that

$$\begin{aligned}|\log|f(z)|| &\leqslant |\log|f_1(z)||+|\log|f_2(z)|| \\ &= O(r_\nu^{K(1+\rho)}/q_\nu)+O(q_\nu r_\nu^{\eta(\rho-1)}) \\ &= O\{r_\nu^{\frac{1}{2}K(1+\rho)-\frac{1}{2}K^2(1-\rho)}\} \\ &= O\left\{\exp\left(\frac{K(1+\rho)-K^2(1-\rho)}{K(1+\rho)+K^2(1-\rho)}\log r\right)\right\}.\end{aligned}$$

We note that

$$\begin{aligned}\rho-\frac{(1+\rho)-K(1-\rho)}{(1+\rho)+K(1-\rho)} &= \frac{\rho(1+\rho)+K\rho(1-\rho)-(1+\rho)+K(1-\rho)}{1+\rho+K(1-\rho)} \\ &= \frac{(K-1)(1-\rho^2)}{1+\rho+K(1-\rho)} \\ &= \frac{(\eta-1)(1-\rho)(1+\rho)}{(K+1)(1+\rho+K(1-\rho))} \\ &> \tfrac{1}{5}(\eta-1)(1-\rho),\end{aligned}$$

in view of (2.4) and (2.5), so that (10.3) holds in (10.1).

Similarly, in the range (10.2) we deduce from (9.1), (9.3), (6.5), (7.1), (7.3), and (8.2) that

$$\begin{aligned}|\log|f(z)|| &= O(r_\nu^{\rho}\log r_\nu) \\ &= o(r^{\rho/(\rho+K(1-\rho))}\log r).\end{aligned}$$

We note that

$$\begin{aligned}\rho-\frac{\rho}{\rho+K(1-\rho)} &= \frac{\rho^2+K\rho(1-\rho)-\rho}{\rho+K(1-\rho)} = \frac{\rho(1-\rho)(K-1)}{\rho+K(1-\rho)} \\ &= \frac{\rho(1-\rho)(\eta-1)}{(K+1)(\rho+K(1-\rho))} > \tfrac{1}{5}(1-\rho)(\eta-1),\end{aligned}$$

in view of (2.4) and (2.5), and so (10.3) holds also in this range and Lemma 10 is proved.

We now map the unit circle $|w| < 1$ onto the sector

$$\tfrac{1}{2}C_{10} < |z| < 2C_{11}, \quad |\arg z| < \tfrac{1}{2}\pi-\tfrac{2}{3}\delta,$$

by a function $z = \varphi(w)$, such that $\varphi(0) = C_{10}$. Let $R = C_{13}$ be the smallest number such that the image of $|w| < R$ under this map includes the sector

$$C_{10} < |z| < C_{11}, \quad |\arg z| < \tfrac{1}{2}\pi-\tfrac{3}{4}\delta.$$

Then $0 < R < 1$. Consider the function

$$F(w) = f\{q_\nu\varphi(w)\}-a.$$

We apply Jensen's theorem to $F(w)$ in $|w| < 1$ and note that in view of Lemma 10 we have, for large ν,

$$\log|F(w)| < q_\nu^{\rho'} \quad \text{in } |w| < 1.$$

Thus Jensen's formula shows that if n is the number of zeros of $F(w)$ in $|w| < R$, we have

$$n\log\frac{1}{R} < q_\nu^{\rho'} + \log\left|\frac{1}{F(0)}\right|$$

$$= q_\nu^{\rho'} - \log|f(C_{10}q_\nu) - a| < q_\nu^{\rho'} + \log\left(\frac{2}{|a|}\right),$$

if ν is large in view of (9.5). Since n exceeds the number of roots of $f(z) = a$ in the region

(10.4) $$C_{10}q_\nu < |z| < C_{11}q_\nu, \quad |\arg z| < \tfrac{1}{2}\pi - \tfrac{3}{4}\delta,$$

we see that the number of zeros of $f(z)-a$ in (10.4) is $O(q_\nu^{\rho'})$.

Similarly, by mapping $|w| < 1$ onto the sector

$$\tfrac{1}{2}C_{12} < |z| < 2C_9, \quad |\arg z| < \tfrac{1}{2}\pi - \tfrac{2}{3}\delta,$$

by $\varphi(w)$, so that $\varphi(0) = C_9$, we deduce that the number of zeros of $f(z)-a$ in

(10.5) $$C_{12}p_\nu < |z| < C_9 p_\nu, \quad |\arg z| < \tfrac{1}{2}\pi - \tfrac{3}{4}\delta,$$

is $O(p_\nu^{\rho'})$. Since all but a finite number of the zeros of $f(z)-a$ in S lie in the regions (10.4) and (10.5) in view of Lemma 9, we deduce that, for every $a \neq 0$,

$$n(r, a, S) = O(r^{\rho'}) \quad \text{as } r \to \infty,$$

so that $k(a, S) \leqslant \rho'$. This completes the proof of Theorem 2.

In conclusion we should like to express our gratitude to Dame Mary Cartwright for her work in discovering the conjecture of Littlewood mentioned in § 1, and for bringing it to our attention, and to Trinity College, Cambridge for allowing us to see this material. We should also like to thank the referee, Professor David Drasin, for his careful reading and many detailed suggestions, which have greatly improved this paper.

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