MEROMORPHIC FUNCTIONS AND THEIR DERIVATIVES

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1. *Introduction*

In the theory of meromorphic functions W. K. Hayman [3] made the following important conjecture. Suppose that F is a family of functions meromorphic in a domain D and that k is a positive integer. If $f(z) \neq 0$ and $f^{(k)}(z) \neq 1$ in D for all f in F, then F is normal in D. Recently Ku Yung-hsing [5] succeeded in proving this. The present paper contains a natural and direct proof of the following theorem.

THEOREM 1. *If k is a positive integer, $f(z)$ is meromorphic in $|z| < 1$, and $f(z) \neq 0$, $f^{(k)}(z) \neq 1$ there, then either $|f(z)| < 1$ or $|f(z)| > C$ uniformly in $|z| < 1/32$, where C is a positive constant which depends only on k.*

Ku's result follows at once from Theorem 1. As another application we derive a result on the existence of a singular direction.

THEOREM 2. *Let $f(z)$ be a function meromorphic in the plane. If*

$$\overline{\lim_{r \to \infty}} \frac{T(r, f)}{(\log r)^3} = \infty, \tag{1}$$

then there is a number θ_0 such that $0 \leqslant \theta_0 < 2\pi$ and for every positive ε and every positive integer k, either $f(z)$ assumes every finite value infinitely often or $f^{(k)}(z)$ assumes every finite value except zero infinitely often in the angle $|\arg z - \theta_0| < \varepsilon$.

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2. *Preliminary lemmas*

LEMMA 1. *Suppose that $f(z)$ is meromorphic in $|z| < R$ $(0 < R \leqslant \infty)$. If $f(0) \neq 0, \infty$, $f^{(k)}(0) \neq 1$ and $f^{(k+1)}(0) \neq 0$, then we have*

$$T(r, f) < \bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - 1}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f)$$

for $0 < r < R$, where

$$S(r, f) = m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{f^{(k+1)}}{f}\right) + m\left(r, \frac{f^{(k+1)}}{f^{(k)} - 1}\right) + \log\left|\frac{f(0)\{f^{(k)}(0) - 1\}}{f^{(k+1)}(0)}\right| + \log 2. \tag{2}$$

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Lemma 1 is substantially due to H. Milloux (see for example [**1**, **2**]), but there is some difference in the error term $S(r, f)$. In fact the identity

$$\frac{1}{f} = \frac{f^{(k)}}{f} - \frac{(f^{(k)}-1)}{f^{(k+1)}} \cdot \frac{f^{(k+1)}}{f}$$

leads to

$$m\left(r, \frac{1}{f}\right) \leqslant m\left(r, \frac{f^{(k)}}{f}\right) + m\left(r, \frac{f^{(k)}-1}{f^{(k+1)}}\right) + m\left(r, \frac{f^{(k+1)}}{f}\right) + \log 2\,. \tag{3}$$

Applying the Jensen–Nevanlinna formula to $m(r, 1/f)$ and $m(r, (f^{(k)}-1)/f^{(k+1)})$, we obtain from (3) that

$$T(r, f) \leqslant N\left(r, \frac{1}{f}\right) + N\left(r, \frac{f^{(k+1)}}{f^{(k)}-1}\right) - N\left(r, \frac{f^{(k)}-1}{f^{(k+1)}}\right) + S(r, f),$$

where $S(r, f)$ is given by (2). Since

$$N\left(r, \frac{f^{(k+1)}}{f^{(k)}-1}\right) - N\left(r, \frac{f^{(k)}-1}{f^{(k+1)}}\right) = \bar{N}(r, f) + N\left(r, \frac{1}{f^{(k)}-1}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right)$$

holds, the assertion of the lemma follows.

LEMMA 2. *Suppose that k is a positive integer and that $f(z)$ is a function meromorphic in $|z| < R$ $(0 < R \leqslant \infty)$ and such that $f(0) \neq 0, \infty$, $f^{(k)}(0) \neq 1$, $f^{(k+1)}(0) \neq 0$ and*

$$(k+1)f^{(k+2)}(0)\{f^{(k)}(0)-1\} - (k+2)\{f^{(k+1)}(0)\}^2 \neq 0\,.$$

Then we have

$$T(r, f) < \left(2 + \frac{1}{k}\right) N\left(r, \frac{1}{f}\right) + \left(2 + \frac{2}{k}\right) \bar{N}\left(r, \frac{1}{f^{(k)}-1}\right) + S(r, f) \tag{4}$$

for $0 < r < R$, where

$$\begin{aligned} S(r, f) = {} & \left(2 + \frac{2}{k}\right) m\left(r, \frac{f^{(k+1)}}{f^{(k)}-1}\right) + \left(2 + \frac{1}{k}\right)\left\{m\left(r, \frac{f^{(k+1)}}{f}\right) + m\left(r, \frac{f^{(k)}}{f}\right)\right\} \\ & + \frac{1}{k} m\left(r, \frac{f^{(k+2)}}{f^{(k+1)}}\right) + 4 + \left(2 + \frac{1}{k}\right) \log\left|\frac{f(0)\{f^{(k)}(0)-1\}}{f^{(k+1)}(0)}\right| \\ & + \frac{1}{k} \log\left|\frac{f^{(k+1)}(0)\{f^{(k)}(0)-1\}}{(k+1)f^{(k+2)}(0)\{f^{(k)}(0)-1\} - (k+2)\{f^{(k+1)}(0)\}^2}\right|. \end{aligned} \tag{5}$$

Lemma 2 is Theorem 1 of Hayman [**1**] except for a slight improvement in the expression for $S(r, f)$, which is important for our applications. The proof is exactly that of [**1**], except for observing that the quantity there denoted by $S_1(r)$ can be expressed by our Lemma 1 in the form (2) instead of the form used by Hayman.

LEMMA 3 (Hiong King-lai [4]). *Suppose that $f(z)$ is meromorphic in $|z| < R$ $(0 < R \leqslant \infty)$ and that k is a positive integer. If $f(0) \neq 0, \infty$ then we have*

$$m\left(r, \frac{f^{(k)}}{f}\right) < C\left(1+\log^+\rho+\log^+\frac{1}{r}+\log^+\frac{1}{\rho-r}+\log^+\log^+\frac{1}{|f(0)|}+\log^+ T(\rho, f)\right)$$

for $0 < r < \rho < R$, where C is a positive constant which depends only on k.

LEMMA 4 [6; pp. 24–25]. *Suppose that $U(r)$ is a non-negative and non-decreasing function in the interval $[R_1, R_2]$ $(0 < R_1 < R_2 < \infty)$, and that a and b are positive constants satisfying $b > (a+2)^2$. If the inequality*

$$U(r) < a\left(\log^+ U(\rho)+\log\frac{\rho}{\rho-r}\right)+b$$

holds for every pair of r, ρ $(R_1 < r < \rho < R_2)$, then we have

$$U(r) < 2a\log\big(R/(R-r)\big)+2b\,.$$

Notation. Throughout the paper C will denote a positive constant which depends at most on the integer k. It will not necessarily be the same constant throughout the course of the argument.

LEMMA 5. *Suppose that f satisfies the assumptions of Lemma 2 and suppose that in addition $f(z) \neq 0$, $f^{(k)}(z) \neq 1$ in $|z| < R$. Then we have*

$$\log M\left(r, \frac{1}{f}\right) < C\frac{R}{R-r}\left(1+B+\log^+\frac{R}{R-r}\right),$$

for $0 < r < R$, where

$$B = \log^+ R+\log^+\frac{1}{R}+\log^+|f(0)|+\log^+|f^{(k)}(0)|+\log^+\frac{1}{|f^{(k+1)}(0)|}$$

$$+\log^+\frac{1}{|(k+1)f^{(k+2)}(0)\{f^{(k)}(0)-1\}-(k+2)\{f^{(k+1)}(0)\}^2|}\,. \tag{6}$$

Proof. In this case $T(r, f) < S(r, f)$ in (4), (5). We estimate the terms of (5). Choosing ρ' and ρ such that $0 < r < \rho' = (\rho+r)/2 < \rho < R$, Nevanlinna's estimate (see for example [2; p. 36]) gives

$$m\left(r, \frac{f^{(k+1)}}{f^{(k)}-1}\right) < C\left\{1+\log^+\rho'+\log^+\frac{1}{r}+\log^+\frac{1}{\rho'-r}\right.$$

$$\left.+\log^+\log^+\frac{1}{|f^{(k)}(0)-1|}+\log^+ T(\rho', f^{(k)})\right\} \tag{7}$$

and

$$m\left(r, \frac{f^{(k+2)}}{f^{(k+1)}}\right) < C\left\{1+\log^+ \rho' + \log^+ \frac{1}{r} + \log^+ \frac{1}{\rho'-r}\right.$$

$$\left. + \log^+ \log^+ \frac{1}{|f^{(k+1)}(0)|} + \log^+ T(\rho', f^{(k+1)})\right\}. \qquad (8)$$

(The usual estimate would give (7) with $\log^+ T(\rho', f^{(k)}-1)$ as its last term, but $|T(\rho', f^{(k)}-1) - T(\rho', f^{(k)})| \leqslant \log 2$.)

For the terms $\log^+ T(\rho', f^{(j)})$ $(j = k, k+1)$, which appear in (7) and (8) we have

$$\log^+ T(\rho', f^{(j)}) \leqslant \log^+\{(j+1)T(\rho', f) + m(\rho', f^{(j)}/f)\}$$

$$< \log^+ T(\rho', f) + m(\rho', f^{(j)}/f) + C.$$

Thus from (4), (5), (7) and (8) we have

$$T(r, f) < C\left\{1+\log^+ \rho' + \log^+ \frac{1}{r} + \log^+ \frac{1}{\rho'-r} + \log^+ T(\rho', f)\right.$$

$$\left. + \log^+ \log^+ \frac{1}{|f^{(k)}(0)-1|} + \log^+ \log^+ \frac{1}{|f^{(k+1)}(0)|}\right\}$$

$$+ \left(2+\frac{1}{k}\right)\log|f(0)| + \left(2+\frac{2}{k}\right)\log|f^{(k)}(0)-1| + 2\log\frac{1}{|f^{(k+1)}(0)|}$$

$$+ \frac{1}{k}\log\frac{1}{|(k+1)f^{(k+2)}(0)\{f^{(k)}(0)-1\} - (k+2)\{f^{(k+1)}(0)\}^2|}$$

$$+ C\left\{m\left(\rho', \frac{f^{(k)}}{f}\right) + m\left(\rho', \frac{f^{(k+1)}}{f}\right)\right\}. \qquad (9)$$

We apply Lemma 3 to the last two terms of (9) with the r, ρ of Lemma 3 equal to ρ' and ρ respectively. Noting the relations between r, ρ', ρ and R we have in the case when $R/2 < r < \rho < R$ that

$$T\left(r, \frac{1}{f}\right) < C\left\{1+\log^+ R + \log^+ \frac{1}{R} + \log^+ \frac{1}{\rho-r} + \log^+ \log^+ \frac{1}{|f(0)|}\right.$$

$$\left. + \log^+ \log^+ \frac{1}{|f^{(k)}(0)-1|} + \log^+ \log^+ \frac{1}{|f^{(k+1)}(0)|} + \log^+ T(\rho, f)\right\}$$

$$+ \left(2+\frac{1}{k}\right)\log|f(0)| + \left(2+\frac{2}{k}\right)\log|f^{(k)}(0)-1| + 2\log\frac{1}{|f^{(k+1)}(0)|}$$

$$+ \frac{1}{k}\log\frac{1}{|(k+1)f^{(k+2)}(0)\{f^{(k)}(0)-1\} - (k+2)\{f^{(k+1)}(0)\}^2|}. \qquad (10)$$

For $\beta > 0$, $0 < x < \infty$ we have

$$\beta \log x + C \log^+ \log^+ (1/x) < \beta \log^+ x + C .$$

Applying this with $x = |f(0)|$ and $x = |f^{(k)}(0) - 1|$, assuming that $R/2 < r < \rho < R$ (so that $\log^+ \dfrac{1}{\rho - r} \leqslant \log \dfrac{\rho}{\rho - r} + \log^+ \dfrac{2}{R}$) and noting that

$$\log^+ T(\rho, f) = \log^+ \{T(\rho, 1/f) + \log |f(0)|\} \leqslant \log^+ T(\rho, 1/f) + \log^+ |f(0)| + 1 ,$$

(10) yields that

$$T\left(r, \frac{1}{f}\right) < C_1(1+B) + C_2 \left\{\log \frac{\rho}{\rho - r} + \log^+ T\left(\rho, \frac{1}{f}\right)\right\}, \tag{11}$$

where B is given by (6). Increasing C_1 so that $C_1 > (C_2+2)^2$, we can then apply Lemma 4 to $T(r, 1/f)$ and deduce that

$$T(r, 1/f) < C\Big(1 + B + \log\big(R/(R-r)\big)\Big), \qquad (R/2 < r < R). \tag{12}$$

For any r such that $0 < r < R$ we have

$$\log M\left(r, \frac{1}{f}\right) \leqslant \frac{R+3r}{R-r} T\left(\frac{r+R}{2}, \frac{1}{f}\right)$$

and by using (12) the proof of the lemma follows.

3. *Proof of Theorem* 1

Suppose that f satisfies the hypotheses of Theorem 1. The conclusions will hold with $C = 1$ unless there are points z', z'' such that $|f(z')| \geqslant 1$, $|f(z'')| \leqslant 1$, $|z'| < 1/32$, $|z''| < 1/32$, and thus by continuity a point z_1 such that

$$|f(z_1)| = 1, \qquad |z_1| < 1/32 . \tag{13}$$

We assume that (13) holds and show that $|f(z)| > C$ uniformly in $|z| < 1/32$. There are two mutually exclusive cases.

Case A. One has

$$\sum_{j=0}^{k+1} |f^{(j)}(z)| \geqslant 1/4 \qquad \text{uniformly in } |z| < 1/8 .$$

It follows that

$$\frac{1}{|f|} \leqslant 4 \sum_{j=0}^{k+1} \left|\frac{f^{(j)}}{f}\right| \qquad (|z| < 1/8),$$

and so if $m(r, z_1, f)$ and $T(r, z_1, f)$ denote $m(r, f(z+z_1))$ and $T(r, f(z+z_1))$ respectively, we have

$$m\left(r, z_1, \frac{1}{f}\right) \leqslant \sum_{j=0}^{k+1} m\left(r, z_1, \frac{f^{(j)}}{f}\right) + \log 4(k+2) \qquad (0 < r < 3/32). \tag{14}$$

Since $N(r, z_1, 1/f) = 0$, applying Lemma 3 to $f(z+z_1)$ yields in (14)

$$T\left(r, z_1, \frac{1}{f}\right) = m\left(r, z_1, \frac{1}{f}\right) \leqslant C\left(1 + \log^+ \frac{1}{\rho - r} + \log^+ T(\rho, z_1, f)\right)$$

for $1/32 < r < \rho < 3/32$. On using Jensen's theorem and noting that $|f(z_1)| = 1$, the last term on the right can be replaced by $\log^+ T(\rho, z_1, 1/f)$. On noting that

$$\log^+ (1/(\rho - r)) \leqslant \log(\rho/(\rho - r)) + \log 32$$

and that C is arbitrary, we can apply Lemma 4 to $T(r, z_1, 1/f)$ in $[1/32, 3/32]$ and obtain

$$T\left(r, z_1, \frac{1}{f}\right) < C\left(1 + \log \frac{3/32}{(3/32) - r}\right),$$

whence $T(5/64, z_1, 1/f) < C$, and

$$\log M(1/32, 1/f) \leqslant \log M(1/16, z_1, 1/f) \leqslant 9T(5/64, z_1, 1/f) < C.$$

Case B. There is a point z_2 such that

$$\sum_{j=0}^{k+1} |f^{(j)}(z_2)| < 1/4, \qquad |z_2| < 1/8. \tag{15}$$

We assert that there exists a point z_0 on the segment $\overline{z_2 z_1}$ such that

$$|f^{(k+2)}(z_0)| \geqslant 1,\ 1/12 < |f^{(k+1)}(z_0)| < 1/2,\ |f^{(k)}(z_0)| < 1/2,\ |f(z_0)| < 1/2. \tag{16}$$

(This technique was also used in our earlier paper [**8**].)

In fact if $|f^{(k+1)}(z)| < 1/4$ on $\overline{z_1 z_2}$ the inequality (15) leads to

$$|f^{(k)}(z)| \leqslant |f^{(k)}(z_2)| + \left| \int_{\overline{z_2 z}} f^{(k+1)}(t)\,dt \right| < \frac{1}{4} + \frac{1}{4}|z_2 - z| < \frac{1}{3},$$

and so successively to

$$|f^{(j)}(z)| < 1/3, \qquad j = k-1, k-2, \ldots, 1, 0;$$

the last of these contradicts the fact that $|f(z_1)| = 1$. Thus there is a point z_3 on $\overline{z_2 z_1}$

such that $|f^{(k+1)}(z_3)| = 1/4$ and $|f^{(k+1)}(z)| < 1/4$ on $\overline{z_2 z_3}$. Clearly

$$|f^{(k)}(z_3)| \leqslant |f^{(k)}(z_2)| + \left| \int_{\overline{z_2 z_3}} f^{(k+1)}(t)dt \right| < 1/3 ,$$

and by similar arguments

$$|f^{(j)}(z_3)| < 1/3 , \qquad j = k-1, ..., 1, 0 .$$

If $|f^{(k+2)}(z_3)| \geqslant 1$, we may take z_3 to be the z_0 in (16).

If $|f^{(k+2)}(z_3)| < 1$, we note that if $|f^{(k+2)}(z)| < 1$ on $\overline{z_3 z_1}$, then on $\overline{z_2 z_1}$

$$|f^{(k+1)}(z)| < \frac{1}{4} + \frac{1}{8} + \frac{1}{32} < \frac{1}{2},$$

and so

$$|f^{(k)}(z)| \leqslant |f^{(k)}(z_2)| + |z_2 - z_1| \text{Max} |f^{(k+1)}(z)| < 1/3 .$$

We then obtain $|f^{(j)}(z)| < 1/3$ on $\overline{z_1 z_2}$ for $j = 0, 1, ..., k$, which contradicts the fact that $|f(z_1)| = 1$. Then there is a point z_4 on $\overline{z_3 z_1}$ such that $|f^{(k+2)}(z_4)| = 1$ and $|f^{(k+2)}(z)| < 1$ on $\overline{z_3 z_4}$. Since $|z_3 - z_4| < |z_1 - z_2| \leqslant (1/32) + (1/8) = 5/32$, we have, for every point of $\overline{z_3 z_4}$,

$$|f^{(k+1)}(z)| \geqslant |f^{(k+1)}(z_3)| - |z_3 - z_4| \operatorname*{Max}_{\overline{z_3 z_4}} |f^{(k+2)}| > 1/12 ,$$

$$|f^{(k+1)}(z)| \leqslant |f^{(k+1)}(z_3)| + |z_3 - z_4| \operatorname*{Max}_{\overline{z_3 z_4}} |f^{(k+2)}| < 1/2 .$$

Thus

$$|f^{(k)}(z_4)| \leqslant |f^{(k)}(z_3)| + |z_3 - z_4| \operatorname*{Max}_{\overline{z_3 z_4}} |f^{(k+1)}| < 1/2 ,$$

and similarly

$$|f^{(j)}(z_4)| < 1/2, \qquad j = 0, 1, ..., k-1 .$$

Thus in this case we may choose $z_0 = z_4$ in (16) and the validity of (16) has been established in all cases.

We now apply Lemma 5 to $f(z)$ in $|z - z_0| < 7/8$. The only condition which needs checking follows from (16):

$$|(k+1)f^{(k+2)}(z_0)\{f^{(k)}(z_0) - 1\} - (k+2)\{f^{(k+1)}(z_0)\}^2| > \frac{k+1}{2} - \frac{k+2}{4} \geqslant \frac{1}{4}.$$

From Lemma 5 we see that

$$\log M(1/2, z_0, 1/f) < C ,$$

and hence

$$\log M(1/32, 1/f) < \log M(1/2, z_0, 1/f) < C .$$

Remark. One may ask why we do not start our work from Hayman's inequality. If we do so and note that the unique difference between Hayman's inequality and

Lemma 2 is the appearance of $m(r, f^{(k+1)}/f^{(k)})$ in the former and $m(r, f^{(k+1)}/f)$ in the latter, then a lemma which is analogous to Lemma 5 except in having B replaced by

$$B' = B + C \log^+ \log^+ \left(1/|f^{(k)}(0)|\right)$$

can be obtained. In order to eliminate the "initial values", we have to find a point z_0 satisfying all the conditions in (16) and $|f^{(k)}(z_0)| > C$. It seems to me that this is impossible. Ku [5] established three lemmas to estimate $m(r, f^{(k+1)}/f^{(k)})$ in which the initial values are

$$\log^+ \log^+ |f(0)| + \log^+ \log^+ \left(1/|f(0)|\right) + \log^+ \log^+ |f(\zeta_0)| + \log^+ \log^+ \left(1/|f^{(k)}(\zeta_0)|\right),$$

where ζ_0 is another point. His proof is ingenious, but not natural.

4. *Proof of Theorem* 2

According to a result of Valiron [7], if $f(z)$ satisfies (1), then there exists a sequence of discs

$$G_j : |z - z_j| < \varepsilon_j |z_j|, \qquad \lim_{j\to\infty} |z_j| = \infty, \qquad \lim_{j\to\infty} \varepsilon_j = 0,$$

such that $f(z)$ takes every complex value n_j times in G_j, with the exception of some values contained in two spherical circles with radius e^{-n_j} provided that $\lim\limits_{j\to\infty} (n_j/\log|z_j|) = \infty$.

Denote by θ_0 an accumulation point of $(\arg z_j,\ j = 1, 2, \ldots)$. It is no loss in generality to suppose that $\arg z_j \to \theta_0$ $(j \to \infty)$. We shall prove that the ray $\arg z = \theta_0$ has the desired property of Theorem 2.

In fact, if it is not true, then there exist a positive number ε, a positive integer k and two finite values a, b $(b \neq 0)$ such that $f(z) \neq a$, $f^{(k)}(z) \neq b$ in the angle $|\arg z - \theta_0| < \varepsilon$.

When j is sufficiently large, the discs

$$G'_j : |z - z_j| < 32\,\varepsilon_j |z_j|$$

are contained in $|\arg z - \theta_0| < \varepsilon$. For every fixed j, the function

$$g_j(t) = \frac{f(z_j + 32\,\varepsilon_j |z_j| t) - a}{b(32\,\varepsilon_j |z_j|)^k}$$

is meromorphic in $|t| < 1$ and $g_j(t) \neq 0$, $g_j^{(k)}(t) \neq 1$ there. Theorem 1 yields that either $|g_j(t)| < 1$ or $|g_j(t)| > C$ in $|t| < 1/32$.

(1) Suppose that $|g_j(t)| < 1$ uniformly in $|t| < 1/32$, that is,

$$|f(z)| < |a| + |b|(32\,\varepsilon_j |z_j|)^k < |z_j|^{k+1}$$

uniformly in G_j, when j is sufficiently large.

Since the spherical distance between $|z_j|^{k+1}$ and ∞ is

$$1/(1+|z_j|^{2(k+1)})^{1/2} > 1/2|z_j|^{k+1},$$

the image of G_j under $w = f(z)$ lies outside the set D of these points w' such that the spherical distance $|w', \infty|$ is less than $(2|z_j|^{k+1})^{-1}$. On the other hand, the image of G_j under $w = f(z)$ covers $|w| < \infty$, apart from two spherical circles with radius e^{-n_j}, where $\lim_{j\to\infty} (n_j/\log|z_j|) = \infty$. Putting $n_j = m_j \log|z_j|$, we have $\lim_{j\to\infty} m_j = \infty$. Thus the values which are not taken by $f(z)$ in G_j can be contained in two spherical circles with radius

$$e^{-n_j} = e^{-m_j \log|z_j|} = 1/|z_j|^{m_j}.$$

Clearly these two circles cannot cover the spherical circles $|w, \infty| < 1/2|z_j|^{k+1}$ and so we derive a contradiction.

(2) Suppose that $|g_j(t)| > C$ uniformly in $|t| < 1/32$.

Now we can suppose that $\varepsilon_j|z_j| > 1$ $(j \to \infty)$, for otherwise we can choose $\varepsilon'_j = \max(\varepsilon_j, 2/|z_j|)$ and replace the discs G_j by the larger discs $|z - z_j| < \varepsilon'_j|z_j|$, which satisfy the same conditions. Thus in $|z| < 1/32$ we have

$$|f(z) - a| > C|b|(32\varepsilon_j|z_j|)^k > (32)^k|b|C.$$

Thus the image of G_j under $w = f(z)$ is entirely disjoint from the fixed disc $|w - a| \leqslant C$. But for large j this disc is not contained in any two spherical circles of radius e^{-n_j}. Thus we have a contradiction and Theorem 2 has been proved.

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