Normal Families and Fix-points of Meromorphic Functions

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1. Introduction

Let $f(z)$ be a meromorphic function in a region D and z_0 a point of D. If $f(z_0) = z_0$, then z_0 is a fix-point of $f(z)$. There are many papers on fix-points of entire and meromorphic functions (cf. [1]–[5]). It seems to me, however, that the connection between the normality of a given family of holomorphic or meromorphic functions and the lack of fix-points of both these functions and their derivatives has not been studied.

The principal aim of this paper is to prove the following theorem.

Theorem 1. *Let $\mathscr{F}$ be a family of meromorphic functions in a region D and k be a positive integer. If, for every function $f(z)$ of $\mathscr{F}$, both $f(z)$ and $f^{(k)}(z)$ (the derivative of order k) have no fix-points in D, then $\mathscr{F}$ is normal there.*

For the proof of Theorem 1, Sections 2 and 3 are devoted to the case of $k = 1$, which is the most important. Then, in Section 4, we give a brief formulation of the case $k \geq 2$.

2. Preliminary Lemmas

Lemma 1. *Suppose that $f(z)$ is meromorphic in $|z| < R$ $(0 < R \leq \infty)$. If $f(0) \neq 0, \infty$; $f'(0) \neq d$ and $df''(0) - f'(0) \neq 0$, then*

$$(2.1)\quad T(r,f) < \bar{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f' - (z + d)}\right) - N\left(r, \frac{1}{(z + d)f'' - f'}\right) + S(r,f)$$

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for $0 < r < R$, *where*

$$S(r,f) = 2m\left(r, \frac{f'}{f}\right) + m\left(r, \frac{f''}{f}\right) + m\left(r, \frac{\{f' - (z+d)\}'}{f' - (z+d)}\right) + m(r, z+d) + 2m\left(r, \frac{1}{z+d}\right) + 3\log 2 + \log\left|\frac{f(0)(f'(0) - d)}{df''(0) - f'(0)}\right|. \tag{2.2}$$

Proof. We start with the identity

$$\frac{1}{f} \equiv \frac{f'}{(z+d)f} - \frac{(z+d)f'' - f'}{(z+d)f} \cdot \frac{f' - (z+d)}{(z+d)f'' - f'}, \tag{2.3}$$

which leads to

$$m\left(r, \frac{1}{f}\right) \le m\left(r, \frac{f'}{(z+d)f}\right) + m\left(r, \frac{(z+d)f'' - f'}{(z+d)f}\right) + m\left(r, \frac{f' - (z+d)}{(z+d)f'' - f'}\right) + \log 2.$$

Applying the Jensen-Nevanlinna formula to

$$m\left(r, \frac{1}{f}\right) \quad \text{and} \quad m\left(r, \frac{f' - (z+d)}{(z+d)f'' - f'}\right),$$

we have

$$m\left(r, \frac{1}{f}\right) = T(r,f) - N\left(r, \frac{1}{f}\right) + \log\frac{1}{|f(0)|}$$

and

$$m\left(r, \frac{f' - (z+d)}{(z+d)f'' - f'}\right) = m\left(r, \frac{(z+d)f'' - f'}{f' - (z+d)}\right) + \left\{N\left(r, \frac{(z+d)f'' - f'}{f' - (z+d)}\right) - N\left(r, \frac{f' - (z+d)}{(z+d)f'' - f'}\right)\right\} + \log\left|\frac{f'(0) - d}{df''(0) - f'(0)}\right|.$$

Since

$$N\left(r, \frac{(z+d)f'' - f'}{f' - (z+d)}\right) - N\left(r, \frac{f' - (z+d)}{(z+d)f'' - f'}\right) = N(r, (z+d)f'' - f') - N(r, f' - (z+d)) + N\left(r, \frac{1}{f' - (z+d)}\right) - N\left(r, \frac{1}{(z+d)f'' - f'}\right)$$

$$\leq \bar{N}(r,f) + N\left(r, \frac{1}{f' - (z+d)}\right) - N\left(r, \frac{1}{(z+d)f'' - f'}\right)$$

and

$$m\left(r, \frac{(z+d)f'' - f'}{f' - (z+d)}\right) = m\left(r, \frac{(z+d)(f''-1) - \{f' - (z+d)\}}{f' - (z+d)}\right)$$

$$\leq m(r, z+d) + m\left(r, \frac{f''-1}{f'-(z+d)}\right) + \log 2,$$

the conclusion of Lemma 1 follows.

Lemma 2. *Let $f(z)$ be as given in Lemma 1 and*

$$g(z) = \frac{\{(z+d)f'' - f'\}^2}{(z+d)^3\{(z+d) - f'\}^3}. \tag{2.4}$$

If $g(0) \neq 0, \infty$; $g'(0) \neq 0$, $d \neq 0$, then we have

$$N_{1)}(r,f) \leq \bar{N}_{(2}(r,f) + \bar{N}\left(r, \frac{1}{f'-(z+d)}\right) + \bar{N}\left(r, \frac{1}{(z+d)f''-f'}\right) \tag{2.5}$$

$$+ m\left(r, \frac{g'}{g}\right) + \log\left|\frac{g(0)}{g'(0)}\right| + 2\log^+ \frac{r}{|d|}$$

where $N_{1)}(r,f)$ denotes the counting function of simple poles of $f(z)$ and $\bar{N}_{(2}(r,f)$ denotes the counting function of multiple poles of $f(z)$, each of them counted only once.

Proof. Suppose $f(z)$ has a simple pole at z_0 and $z_0 \neq -d$. Thus

$$f(z) = \frac{a}{z-z_0} + O(1), \qquad (a \neq 0)$$

$$f'(z) = \frac{-a}{(z-z_0)^2} + O(1)$$

and

$$f''(z) = \frac{2a}{(z-z_0)^3} + O(1)$$

in $\Omega(z_0)$, a small neighborhood of z_0.

Since

$$z + d = (z_0 + d) + (z - z_0),$$

an elementary calculation gives

$$g(z) = \frac{\dfrac{4a^2(z_0+d)^2}{(z-z_0)^6} + \dfrac{12a^2(z_0+d)}{(z-z_0)^5} + O\left(\dfrac{1}{(z-z_0)^4}\right)}{\dfrac{a^3(z_0+d)^3}{(z-z_0)^6} + \dfrac{3a^3(z_0+d)^2}{(z-z_0)^5} + O\left(\dfrac{1}{(z-z_0)^4}\right)} = \frac{4}{a(z_0+d)}\{1 + O((z-z_0)^2)\}$$

in $\Omega(z_0)$. This means z_0 is neither a zero nor a pole of $g(z)$, but z_0 must be a zero of $g'(z)$. Thus

$$N_{1)}(r,f) \le N_0\left(r, \frac{1}{g'}\right) + \log^+ \frac{r}{|d|}, \tag{2.6}$$

where $N_0(r,1/g')$ denotes the counting function of zeros of $g'(z)$ which are not zeros of $g(z)$.

On the other hand, Jensen's formula gives

$$\begin{aligned} m\left(r, \frac{g'}{g}\right) - m\left(r, \frac{g}{g'}\right) - \log\left|\frac{g'(0)}{g(0)}\right| &= N\left(r, \frac{g}{g'}\right) - N\left(r, \frac{g'}{g}\right) \\ &= N(r,g) - N(r,g') + N\left(r, \frac{1}{g'}\right) - N\left(r, \frac{1}{g}\right) \\ &= -\bar{N}(r,g) + N_0\left(r, \frac{1}{g'}\right) - \bar{N}\left(r, \frac{1}{g}\right). \end{aligned}$$

It follows that

$$N_0\left(r, \frac{1}{g'}\right) \le \bar{N}(r,g) + \bar{N}\left(r, \frac{1}{g}\right) + m\left(r, \frac{g'}{g}\right) + \log\left|\frac{g(0)}{g'(0)}\right|. \tag{2.7}$$

From the expression of $g(z)$, it is clear that any zero or pole of $g(z)$ can only occur at zeros of $f'(z) - (z+d)$, $z = -d$, multiple poles of $f(z)$ and zeros of $(z+d)f'' - f'$. Therefore

$$\begin{aligned} \bar{N}(r,g) + \bar{N}\left(r, \frac{1}{g}\right) &\le \bar{N}_{(2}(r,f) \\ &+ \bar{N}\left(r, \frac{1}{f' - (z+d)}\right) + \bar{N}\left(r, \frac{1}{(z+d)f'' - f'}\right) + \log^+ \frac{r}{|d|}. \end{aligned} \tag{2.8}$$

Comparing (2.6), (2.7) and (2.8), the inequality (2.5) follows.

Lemma 3. *Suppose that $f(z)$ is meromorphic in $|z| < R$ $(0 < R \le \infty)$. If $f(0) \ne 0,\infty$; $f'(0) \ne d$; $df''(0) - f'(0) \ne 0$, $d \ne 0$, and*

$$2f'''(0)d^2(f'(0)-d) + 3f'(0)^2 - 3d^2f''(0)^2 + 6d^2f''(0) - 6df'(0) \ne 0,$$

then we have

$$(2.9)\qquad T(r,f) < 3N\left(r,\frac{1}{f}\right) + 4N\left(r,\frac{1}{f'-(z+d)}\right) + S_1(r,f)$$

for $0 < r < R$, *where*

$$(2.10)\quad S_1(r,f) = 6m\left(r,\frac{f'}{f}\right) + 3m\left(r,\frac{f''}{f}\right) + 4m\left(r,\frac{\{f'-(z+d)\}'}{f'-(z+d)}\right)$$
$$+ m\left(r,\frac{\{(z+d)f''-f'\}'}{(z+d)f''-f'}\right) + 3m(r,z+d) + 7m\left(r,\frac{1}{z+d}\right)$$
$$+ \log|d| + 2\log^+\frac{r}{|d|}$$
$$+ 10\log 2 + 3\log 3 + 3\log|f(0)| + 4\log|f'(0)-d|$$
$$+ 2\log\frac{1}{|df''(0)-f'(0)|}$$
$$+ \log\frac{1}{|2f'''(0)d^2(f'(0)-d) + 3f'(0)^2 - 3d^2f''(0)^2 + 6d^2f''(0) - 6df'(0)|}.$$

Proof. Comparing the fact that

$$\bar{N}_{(2}(r,f) + \bar{N}(r,f) \le N(r,f) \le T(r,f)$$

and Lemma 1, we obtain

$$(2.11)\quad \bar{N}_{(2}(r,f) \le N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f'-(z+d)}\right)$$
$$- N\left(r,\frac{1}{(z+d)f''-f'}\right) + S(r,f),$$

where $S(r,f)$ is given by (2.2).

From Lemma 2 and (2.11), we have

$$\bar{N}(r,f) = N_{1)}(r,f) + \bar{N}_{(2}(r,f)$$
$$\le 2N\left(r,\frac{1}{f}\right) + 3N\left(r,\frac{1}{f'-(z+d)}\right) - N\left(r,\frac{1}{(z+d)f''-f'}\right)$$
$$+ 2\log r + m\left(r,\frac{g'}{g}\right) + \log\left|\frac{g(0)}{g'(0)}\right| + 2S(r,f).$$

Substituting this inequality into (2.1) and noting

$$\frac{g'(z)}{g(z)} = \frac{2(z+d)f'''}{(z+d)f''-f'} - \frac{3}{z+d} - \frac{3(1-f'')}{(z+d)-f'},$$

$$m\left(r,\frac{g'}{g}\right) \le m\left(r,\frac{\{(z+d)f''-f'\}'}{(z+d)f''-f'}\right) + m\left(r,\frac{1}{z+d}\right) + m\left(r,\frac{\{f'-(z+d)\}'}{f'-(z+d)}\right) + 3\log 3 + \log 2$$

and

$$\log\left|\frac{g(0)}{g'(0)}\right| = \log\left|\frac{d(df''(0)-f'(0))(d-f'(0))}{2f'''(0)d^2(f'(0)-d)+3f'(0)^2-3d^2f''(0)^2-6df'(0)+6d^2f''(0)}\right|,$$

Lemma 3 follows.

Using the procedure similar to [6] and applying Nevanlinna's fundamental lemma and its extension (cf. [6]) to the first four terms in $S_1(r,f)$, we obtain the following lemma.

Lemma 4. *Suppose that $f(z)$ satisfies the assumptions of Lemma 3 with $R < \infty$ and suppose that in addition $f \neq 0$ and $f' \neq z + d$ $(d \neq 0)$ in $|z| < R$.*
Then we have

$$\log M\left(r,\frac{1}{f}\right) < C\frac{R}{R-r}\left(1 + B + \log\frac{R}{R-r}\right) \tag{2.12}$$

for $0 < r < R$, where C is a positive numerical constant and

$$B = \log^+ R + \log^+\frac{1}{R} + \log^+|f(0)| + \log^+|f'(0)| + \log^+|d| + \log^+\frac{1}{|d|} + \log^+\frac{1}{|df''(0)-f'(0)|} + \log^+\frac{1}{|2f'''(0)d^2(f'(0)-d)+3f'(0)^2-3d^2f''(0)^2-6df'(0)+6d^2f''(0)|}. \tag{2.13}$$

When $d = 0$, (2.2) and (2.5) can be replaced by

$$S'(r,f) = 2m\left(r,\frac{f'}{f}\right) + m\left(r,\frac{f''}{f}\right) + m\left(r,\frac{(f'-z)'}{f'-z)}\right) + m(r,z) + 2m\left(r,\frac{1}{z}\right) + 3\log 2 + \log|f(0)| \tag{2.2$'$}$$

and

$$(2.5)' \quad N_{1)}(r,f) \le \bar{N}_{(2}(r,f) + \bar{N}\left(r, \frac{1}{f' - z}\right) + \bar{N}\left(r, \frac{1}{zf'' - f'}\right) + m\left(r, \frac{g'}{g}\right) + \log |C_\lambda| + 2\log^+ r$$

respectively, where C_λ is the first nonzero term of Taylor series of $g(z)/g'(z)$ in the neighborhood of the origin. Since

$$\frac{g(z)}{g'(z)} = \frac{(zf'' - f')(z - f')}{2z^2 f''(z - f') - 3(z - f')(zf'' - f') - 3z(1 - f'')(zf'' - f')} z$$
$$= -\frac{1}{3} z + O(z^2)$$

in the neighborhood of the origin,

$$(2.14) \quad m\left(r, \frac{g'}{g}\right) \le m\left(r, \frac{(zf'' - f')'}{zf'' - f'}\right) + m\left(r, \frac{(f' - z)'}{f' - z}\right) + m\left(r, \frac{1}{z}\right) + 3\log 3 + \log 2$$

and using Nevanlinna's lemma for the estimate of terms $m(r,f'/f)$, $m(r,f''/f)$, $m(r,(f' - z)'/(f' - z))$ and $m(r,(zf'' - f')'/(zf'' - f'))$ which appear in (2.2)′ and (2.14), (2.12) remains true with

$$(2.13)' \qquad B' = \log^+ R + \log^+ \frac{1}{R} + \log^+ |f(0)| + \log^+ \frac{1}{|f'(0)|}.$$

3. Principal Results

Theorem 2. *If $f(z)$ is meromorphic in $|z| < R$ $(R \le 1)$ and $f(z) \ne 0$, $f'(z) \ne z + d$ there, then either $|f(z)| < 1$ or $|f(z)| > C(R,d)$ uniformly in $|z| < R_1$, where $C(R,d)$ is a positive constant depending only on R and d, and*

$$(3.1) \qquad R_1 = \begin{cases} \dfrac{R}{256} \min(1,|d|), & \text{if } d \ne 0, \\ \dfrac{R}{256}, & \text{if } d = 0. \end{cases}$$

Proof. We suppose first that $d \ne 0$. If neither $|f(z)| < 1$ nor $|f(z)| > 1$ uniformly in $|z| < R_1$, then there are two points z' and z'' such that $|f(z')| \ge 1$, $|f(z'')| \le 1$, $|z'| < R_1$ and $|z''| < R_1$. Thus by continuity a point z_1 must exist such that

$$(3.2) \qquad |f(z_1)| = 1, \qquad |z_1| < R_1.$$

We distinguish two cases which are mutually exclusive.

Case A. One has

$$\sum_{j=0}^{2} |f^{(j)}(z)| \geq \frac{\min(1,|d|)}{8} \qquad \text{uniformly in } |z| < 4R_1. \tag{3.3}$$

It follows that

$$\frac{1}{|f|} \leq \frac{8}{\min(1,|d|)} \sum_{j=0}^{2} \left|\frac{f^{(j)}}{f}\right| \qquad (|z| < 4R_1)$$

and so if $m(r,z_1,f)$ and $T(r,z_1,f)$ denote $m(r,f(z + z_1))$ and $T(r,f(z + z_1))$ respectively, we have

$$m\left(r,z_1,\frac{1}{f}\right) \leq \sum_{j=0}^{2} m\left(r,z_1\frac{f^{(j)}}{f}\right) + \log\frac{24}{\min(1,|d|)}, \qquad (0 < r < 3R_1). \tag{3.4}$$

Since $N(r,z_1,1/f) = 0$, applying Nevanlinna's fundamental lemma and its extension to $f(z + z_1)$ yields in (3.4)

$$T\left(r,z_1,\frac{1}{f}\right) < C\left\{1 + \log\frac{1}{R_1} + \log^+\frac{1}{|d|} + \log^+\frac{1}{\rho - r} + \log^+T(\rho,z_1,f)\right\}$$

for $R_1 < r < \rho < 3R_1$. Noting that $T(\rho,z_1,f) = T(\rho,z_1,1/f)$ and using the improved form of Bureau's lemma (cf. [6]), we obtain

$$T\left(r,z_1,\frac{1}{f}\right) < C\left\{1 + \log\frac{1}{R_1} + \log^+\frac{1}{|d|} + \log\frac{3R_1}{3R_1 - r}\right\}, \qquad (R_1 < r < 3R_1).$$

Therefore

$$\log M\left(R_1,\frac{1}{f}\right) \leq \log M\left(2R_1,z_1,\frac{1}{f}\right) \leq 9T\left(\frac{5}{2}R_1,z_1,\frac{1}{f}\right) < C(R,d).$$

Case B. There exists a point z_2 such that

$$\sum_{j=0}^{2} |f^{(j)}(z_2)| < \frac{\min(1,|d|)}{8}, \qquad |z_2| < 4R_1. \tag{3.5}$$

We claim that there exists a point z_0 on the segment $\overline{z_2z_1}$ such that

$$|f'''(z_0)| \geq \frac{8}{\min(1,|d|)}, \qquad \frac{1}{2} < |f''(z_0)| < 1,$$
$$|f'(z_0)| < \frac{\min(1,|d|)}{4}, \qquad |f(z_0)| < \frac{1}{4}. \tag{3.6}$$

In fact, if $|f''(z)| < 3/4$ uniformly on $\overline{z_2z_1}$, then the inequality (3.5) leads to

$$|f'(z)| \leq |f'(z_2)| + \left(\max_{\zeta\in\overline{z_2z}}|f''(\zeta)|\right)|z_2 - z| < \frac{5}{32}\min(1,|d|)$$

for any point z on $\overline{z_2 z_1}$. Thus

$$|f(z_1)| \leq |f(z_2)| + \left(\max_{\zeta \in \overline{z_2 z_1}} |f'(\zeta)|\right) |z_2 - z_1| < \frac{5}{32}.$$

This contradicts the fact that $|f(z_1)| = 1$. Consequently, there exists a point z_3 on $\overline{z_2 z_1}$ such that $|f''(z_3)| = 3/4$ and $|f''(z)| < 3/4$ uniformly in $\overline{z_2 z_3}$. Clearly, $|f'(z_3)| < 5/32 \min(1,|d|)$ and $|f(z_3)| < 5/32$.

If $|f'''(z_3)| \geq 8/\min(1,|d|)$, we may choose z_3 to be the point z_0 in (3.6).

If $|f'''(z_3)| < 8/\min(1,|d|)$, we claim that $|f'''(z)| < 8/\min(1,|d|)$ cannot hold uniformly on $\overline{z_3 z_1}$. In fact, the opposite case implies

$$|f''(z)| \leq |f''(z_3)| + \left(\max_{\zeta \in \overline{z_3 z}} |f'''(\zeta)|\right) |z_3 - z| < 1$$

and

$$|f'(z)| \leq |f'(z_3)| + \left(\max_{\zeta \in \overline{z_3 z}} |f''(\zeta)|\right) |z_3 - z| < \frac{3}{16}$$

for any point z on $\overline{z_3 z_1}$. We then obtain

$$|f(z_1)| \leq |f(z_3)| + \left(\max_{\zeta \in \overline{z_3 z_1}} |f'(z)|\right) |z_3 - z_1| < \frac{3}{16}$$

which contradicts the fact that $|f(z_1)| = 1$. Thus there is a point z_4 on $\overline{z_3 z_1}$ such that $|f'''(z_4)| = 8/\min(1,|d|)$ and $|f'''(z)| < 8/\min(1,|d|)$ uniformly in $\overline{z_3 z_4}$. It is clear that for every point z on $\overline{z_3 z_4}$,

$$|f''(z)| \geq |f''(z_3)| - \left(\max_{\zeta \in \overline{z_3 z}} |f'''(\zeta)|\right) |z_3 - z| > \frac{1}{2}$$

and

$$|f''(z)| \leq |f''(z_3)| + \left(\max_{\zeta \in \overline{z_3 z}} |f'''(\zeta)|\right) |z_3 - z| < 1.$$

Thus

$$|f'(z)| \leq |f'(z_3)| + \left(\max_{\zeta \in \overline{z_3 z}} |f''(\zeta)|\right) |z_3 - z| < \frac{1}{4} \min(1,|d|)$$

and

$$|f(z)| \leq |f(z_3)| + \left(\max_{\zeta \in \overline{z_3 z}} |f'(\zeta)|\right) |z_3 - z| < \frac{1}{4}$$

for every point z on $\overline{z_3 z_4}$. Thus in this case we may choose $z_0 = z_4$ in (3.6) and the validity of (3.6) has been established in all cases.

We now apply Lemma 4 to $f(z)$ in $|z - z_0| < (63/64)R$. Since

$$|f(z_0)| < \frac{1}{4}, \qquad |f'(z_0)| < \frac{\min(1,|d|)}{4},$$

$$\frac{1}{|df''(z_0) - f'(z_0)|} \le \frac{1}{||d||f''(z_0)| - |f'(z_0)||} \le \frac{4}{|d|}$$

and

$$\frac{1}{|2f'''(z_0)d^2(f'(z_0) - d) + 3f'(z_0)^2 - 3d^2f''(z_0)^2 + 6d^2f''(z_0) - 6df'(z_0)|}$$

$$\le \frac{1}{|2|f'''(z_0)||d|^2(|d| - |f'(z_0)|) - 3|f'(z_0)|^2 - 3|d|^2|f''(z_0)|^2 - 6|d|^2|f''(z_0)| - 6|d||f'(z_0)||}$$

$$\le \max\left(1, \frac{1}{|d|^2}\right),$$

we have

$$\log M\left(\frac{R}{2}, z_0, \frac{1}{f}\right) < C(R,d)$$

and hence

$$\log M\left(\frac{R}{256}, \frac{1}{f}\right) < \log M\left(\frac{R}{2}, z_0, \frac{1}{f}\right) < C(R,d).$$

Finally we consider the case of $d = 0$.

If $|f(z)| > 1/2$ holds uniformly in $|z| < R_1$, then the conclusion of Theorem 2 is also true. Otherwise there is a point z_1' such that

$$|f(z_1')| \le \frac{1}{2}, \qquad |z_1'| < R_1.$$

When $|f'(z)| < 1$ holds uniformly in $|z - z_1'| < 2R_1$, we have

$$|f(z)| \le |f(z_1')| + \left(\max_{t \in \overline{z_1'z}} f'(t)\right)|z - z_1'| < 1$$

in $|z - z_1'| < 2R_1$. Thus

$$M(R_1, f) \le M(2R_1, z_1', f) < 1.$$

In the opposite case, there is a point z_2' in $|z - z_1'| < 2R_1$ such that $|f'(z_2')| \ge 1$ and that $|f'(z)| < 1$ in $\overline{z_1'z_2'}$. (If $|f'(z_1')| \ge 1$, then we choose the point z_1' as z_2'.) Thus $|f(z_2')| < 1$.

We now apply Lemma 4 with (2.13)′ to $f(z)$ in $|z - z_2'| < (127/128)R$. It follows that

$$\log M\left(\frac{R}{2},z_2',\frac{1}{f}\right) < C(R)$$

and hence

$$\log M\left(\frac{R}{256},\frac{1}{f}\right) \leq \log M\left(\frac{R}{2},z_2',\frac{1}{f}\right) < C(R).$$

Theorem 3. *Let $\mathcal{F}$ be a family of meromorphic functions in a region D. If, for every function $f(z)$ of $\mathcal{F}$, both $f(z)$ and $f'(z)$ have no fix-points in D, then $\mathcal{F}$ is normal there.*

In fact, for an arbitrary point z_0 of D, there is a positive number δ such that the disk $|z - z_0| < \delta$ is contained in D. Now we set $R = \min(1,\delta)$ and consider another family $\mathcal{G}$ which consists of functions

$$g(z) = f(z_0 + z) - (z_0 + z), \qquad \forall f \in \mathcal{F}.$$

Clearly, every $g(z)$ is meromorphic in $|z| < R$ and $g(z) \neq 0$, $g'(z) \neq z + (z_0 - 1)$ there. According to Theorem 2, $\mathcal{G}$ is normal at z_0. Thus $\mathcal{F}$ is also normal at z_0.

4. Case of $k \geq 2$

The preceding results can be generalized from $f'(z)$ to $f^{(k)}(z)$ ($k \geq 2$). We formulate here only the lemmas and the theorems, since the proofs are similar.

Lemma 1'. *Let $f(z)$ be meromorphic in $|z| < R$ ($0 < R \leq \infty$). If $f(0) \neq 0$, ∞; $f^{(k)}(0) \neq d$ and $df^{(k+1)}(0) - f^{(k)}(0) \neq 0$, then we have*

$$T(r,f) < \bar{N}(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{f^{(k)} - (z + d)}\right) - N\left(r,\frac{1}{(z + d)f^{(k+1)} - f^{(k)}}\right) + S(r,f)$$

for $0 < r < R$, where

$$S(r,f) = 2m\left(r,\frac{f^{(k)}}{f}\right) + m\left(r,\frac{f^{(k+1)}}{f}\right) + m\left(r,\frac{f^{(k+1)} - 1}{f^{(k)} - (z + d)}\right) + m(r,z + d) + 2m\left(r,\frac{1}{z + d}\right) + 3\log 2 + \log\left|\frac{f(0)(f^{(k)}(0) - d)}{df^{(k+1)}(0) - f^{(k)}(0)}\right|.$$

Lemma 2'. *Let $f(z)$ be given by Lemma 1' and*

$$g(z) = \frac{\{(z + d)f^{(k+1)} - f^{(k)}\}^{k+1}}{(z + d)^{k+2}\{(z + d) - f^{(k)}\}^{k+2}}.$$

If $g(0) \neq 0,\infty$; $g'(0) \neq 0$, $d \neq 0$, then we have

$$N_{1)}(r,f) \le \bar{N}_{(2}(r,f) + \bar{N}\left(r, \frac{1}{f^{(k)} - (z+d)}\right) + \bar{N}\left(r, \frac{1}{(z+d)f^{(k+1)} - f^{(k)}}\right) + m\left(r, \frac{g'}{g}\right) + \log\left|\frac{g(0)}{g'(0)}\right| + 2\log^+ \frac{r}{|d|}.$$

Lemma 3'. *Suppose that $f(z)$ is meromorphic in $|z| < R$ $(0 < R \le \infty)$. If $f(0) \ne 0, \infty$; $f^{(k)}(0) \ne d$; $df^{(k+1)}(0) - f^{(k)}(0) \ne 0$, $d \ne 0$ and*

$$2f^{(k+2)}(0)d^2(f^{(k)}(0) - d) + 3f^{(k)}(0)^2 - 3d^2 f^{(k+1)}(0)^2 + 6d^2 f^{(k+1)}(0) - 6df^{(k)}(0) \ne 0,$$

then we have

$$T(r,f) < 3N\left(r, \frac{1}{f}\right) + 4N\left(r, \frac{1}{f^{(k)} - (z+d)}\right) + S(r,f)$$

for $0 < r < R$, where

$$S(r,f) = 6m\left(r, \frac{f^{(k)}}{f}\right) + 3m\left(r, \frac{f^{(k+1)}}{f}\right) + 4m\left(r, \frac{f^{(k+1)} - 1}{f^{(k)} - (z+d)}\right)$$
$$+ m\left(r, \frac{\{(z+d)f^{(k+1)} - f^{(k)}\}'}{(z+d)f^{(k+1)} - f^{(k)}}\right) + 3m(r, z+d) + 7m\left(r, \frac{1}{z+d}\right) + 2\log^+ \frac{r}{|d|}$$
$$+ \log^+|d| + 9\log 2 + \log 3 + \log(k+1) + 2\log(k+2)$$
$$+ 3\log|f(0)| + 4\log|f^{(k)}(0) - d| + 2\log \frac{1}{|df^{(k+1)}(0) - f^{(k)}(0)|}$$
$$+ \log \frac{1}{|2f^{(k+2)}(0)d^2(f^{(k)}(0) - d) + 3f^{(k)}(0)^2 - 3d^2 f^{(k+1)}(0)^2 + 6d^2 f^{(k+1)}(0) - 6df^{(k)}(0)|}.$$

Lemma 4'. *Suppose that $f(z)$ satisfies the assumptions of Lemma 3' with $R < \infty$ and suppose that in addition $f \ne 0$, $f^{(k)} \ne z + d(d \ne 0)$ in $|z| < R$. Then we have*

$$\log M\left(r, \frac{1}{f}\right) < C_k \frac{R}{R-r}\left(1 + B + \log \frac{R}{R-r}\right)$$

for $0 < r < R$, where

$$B = \log^+ R + \log^+ \frac{1}{R} + \log^+|d| + \log^+ \frac{1}{|d|} + \log^+|f(0)|$$
$$+ \log^+ \frac{1}{|f^{(k)}(0)|} + \log^+ \frac{1}{|df^{(k+1)}(0) - f^{(k)}(0)|}$$
$$+ \log^+ \frac{1}{|2f^{(k+2)}(0)d^2(f^{(k)}(0) - d) + 3f^{(k)}(0)^2 - 3d^2 f^{(k+1)}(0)^2 + 6d^2 f^{(k+1)}(0) - 6df^{(k)}(0)|}.$$

when $d = 0$, *B can be replaced by*

$$B' = \log^+ R + \log^+ \frac{1}{R} + \log^+|f(0)| + \log^+|f^{(k)}(0).$$

Theorem 2'. *If* $f(z)$ *is meromorphic in* $|z| < R$ $(R \leq 1)$ *and* $f \neq 0$, $f^{(k)} \neq z + d$ *there, then either* $|f| < 1$ or $|f| > C(k,R,d)$ *uniformly in* $|z| < R_1$, *where* $C(k,R,d)$ *is a positive constant depending only on k, R and d, and* R_1 *is given by* (3.1).

Theorem 3'. *Let* $\mathcal{F}$ *be a family of meromorphic functions in a region D. If for every function* $f(z)$ *of* $\mathcal{F}$, *both* $f(z)$ *and* $f^{(k)}(z)$ *have no fix-points in D, then* $\mathcal{F}$ *is normal in D.*

Combining Theorem 3 and Theorem 3', we obtain Theorem 1.

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