PRECISE ESTIMATE OF TOTAL DEFICIENCY OF MEROMORPHIC DERIVATIVES

By

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Abstract. Let $f(z)$ be a transcendental meromorphic function in the finite plane and k be a positive integer. Then we have

$$\sum_{a \in \mathbf{C}} \delta(a, f^{(k)}) \leqq \frac{2k+2}{2k+1}.$$

Moreover, if the order of $f(z)$ is finite, then we also have

$$\sum_{a \in \hat{\mathbf{C}}} \delta(a, f^{(k)}) \leqq 2 - \frac{2k(1-\Theta(\infty, f))}{1+k(1-\Theta(\infty, f))},$$

where $\delta(a, f^{(k)})$ denotes the deficiency of the value a with respect to $f^{(k)}$ and $\Theta(\infty, f)$ is the ramification index of ∞ with respect to f.

1. Introduction

Suppose that $f(z)$ is a transcendental meromorphic function in the finite plane and a is a complex value which may be infinity. By R. Nevanlinna [16], [8], if

$$\delta(a, f) = \liminf_{r \to \infty} \frac{m(r, 1/(f-a))}{T(r, f)}$$

$$= 1 - \limsup_{r \to \infty} \frac{N(r, 1/(f-a))}{T(r, f)}$$

is positive, a is called a deficient value of $f(z)$ and $\delta(a, f)$ is its deficiency. (When $a = \infty$, $m(r, 1/(f-a))$ and $N(r, 1/(f-a))$ in the definition of $\delta(a, f)$ should be replaced by $m(r, f)$ and $N(r, f)$ respectively.) The most important and classical result is that the set of all deficient values of $f(z)$ is at most countable and the total deficiency does not exceed two (deficient relation [6], [8]). The upper bound of two is sharp in general.

When the order of $f(z)$ is less than 1, Edrei [3] obtained a precise estimate for the total deficiency by using the spread relation proved by Baernstein [1]. The deficiency problem, however, is still open for meromorphic functions of order bigger than 1, although a suitable bound has been suggested by Drasin and Weitsman [2].

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Now we discuss the precise estimate of the total deficiency, not for the function $f(z)$ itself, but for its derivatives.

Let k be a positive integer. Hayman [5] pointed out that the inequality

$$\sum_{a\in\mathbf{C}} \delta(a, f^{(k)}) \leqq \frac{k+2}{k+1}$$

holds for any transcendental meromorphic function $f(z)$. In 1971, Mues [7] improved this result to

$$\sum_{a\in\mathbf{C}} \delta(a, f^{(k)}) \leqq \frac{k^2+5k+4}{k^2+4k+2}.$$

In this paper, we shall prove

Theorem 1. *Let $f(z)$ be a transcendental meromorphic function in the finite plane and k be a positive integer. Then we have*

$$\sum_{a\in\mathbf{C}} \delta(a, f^{(k)}) \leqq \frac{2k+2}{2k+1}.$$

It is clear that for any positive integer k, we always have

$$\frac{2k+2}{2k+1} < \frac{k^2+5k+4}{k^2+4k+2} < \frac{k+2}{k+1}$$

and

$$\frac{k^2+5k+4}{k^2+4k+2} - \frac{2k+2}{2k+1} > \frac{k+2}{k+1} - \frac{k^2+5k+4}{k^2+4k+2}.$$

Although Theorem 1 gives a much better estimate for $\Sigma_{a\in\mathbf{C}}\, \delta(a, f^{(k)})$, it does not include $\delta(\infty, f^{(k)})$. For this reason, we prove another estimate.

Theorem 2. *Let $f(z)$ be a transcendental meromorphic function of finite order in the finite plane and k be a positive integer. Then we have*

$$\sum_{a\in\hat{\mathbf{C}}} \delta(a, f^{(k)}) \leqq 2 - \frac{2k(1-\Theta(\infty, f))}{1+k(1-\Theta(\infty, f))}$$

where $\Theta(\infty, f)$ is the ramification index of ∞ with respect to f, defined by

$$\Theta(\infty, f) = 1 - \limsup_{r\to\infty} \frac{\bar{N}(r, f)}{T(r, f)}.$$

Finally we will prove a theorem on uniqueness.

2. A lemma

In order to prove Theorem 1, we need the following lemma which is a rewritten form of a lemma due to Frank and Weissenborn [4].

Lemma 1. *Suppose that $f(z)$ is a transcendental meromorphic function. Given any positive number ε, we have*

$$(1)\qquad N\left(r, \frac{1}{f^{(k+1)}}\right) > (k+1)\bar{N}(r,f) - N(r,f) - \varepsilon T(r, f^{(k)}) - S(r, f^{(k)}),$$

where

$$S(r, f^{(k)}) = O\{\log(rT(r, f^{(k)}))\},$$

except for r in a set with finite linear measure.

In fact, according to Frank and Weissenborn [4], we have

$$k\bar{N}(r,f) < N\left(r, \frac{1}{f^{(k+1)}}\right) + (N(r,f) - \bar{N}(r,f)) + \frac{\varepsilon}{3} N\left(r, \frac{1}{f^{(k+1)}}\right)$$
$$+ \frac{\varepsilon}{3}(N(r,f) - \bar{N}(r,f)) + m\left(r, \frac{W}{(f^{(k+1)})^{l+1}}\right),$$

where

$$W(z) = W(1, z, z^2, \ldots, z^{k+l}, f(z), zf(z), \ldots, z^l f(z))$$

denotes the Wronskian and l is a positive integer such that $l > 3(k+1)/\varepsilon$.

Noting

$$N\left(r, \frac{1}{f^{(k+1)}}\right) \leqq T(r, f^{(k+1)}) + O(1)$$
$$\leqq 2T(r, f^{(k)}) + m\left(r, \frac{f^{(k+1)}}{f^{(k)}}\right) + O(1),$$
$$N(r,f) \leqq T(r, f^{(k)})$$

and

$$\frac{W(z)}{(f^{(k+1)})^{l+1}} \equiv \frac{W(1, z, z^2, \ldots, z^{k+l}) \cdot W(f^{(k+l+1)}, (zf)^{(k+l+1)}, \ldots, (z^l f)^{(k+l+1)})}{(f^{(k+1)})^{l+1}}$$
$$\equiv W(1, z, z^2, \ldots, z^{k+l}) \cdot W\left(\frac{f^{(k+l+1)}}{f^{(k+1)}}, \frac{(zf)^{(k+l+1)}}{f^{(k+1)}}, \ldots, \frac{(z^l f)^{(k+l+1)}}{f^{(k+1)}}\right),$$

the inequality (1) follows immediately.

3. Case of two terms in Theorem 1

At first, we shall show [9]

$$\delta(a_1, f^{(k)}) + \delta(a_2, f^{(k)}) \leqq \frac{2k+2}{2k+1}, \tag{2}$$

where a_1 and a_2 are two finite distinct complex values.

We apply the Nevanlinna Second Fundamental Inequality to $f^{(k)}(z)$ and three complex values a_1, a_2 and ∞:

$$\begin{aligned} T(r, f^{(k)}) < \bar{N}(r, f) + N\left(r, \frac{1}{f^{(k)} - a_1}\right) + N\left(r, \frac{1}{f^{(k)} - a_2}\right) \\ - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f^{(k)}). \end{aligned} \tag{3}$$

Substituting (1) in (3) and noting

$$N(r, f^{(k)}) \geqq N(r, f) + k\bar{N}(r, f),$$

we obtain

$$\bar{N}(r, f) < \frac{1}{2k}\left\{N\left(r, \frac{1}{f^{(k)} - a_1}\right) + N\left(r, \frac{1}{f^{(k)} - a_2}\right)\right\} + \varepsilon T(r, f^{(k)}) + S(r, f^{(k)}). \tag{4}$$

Combining (3) and (4), we have

$$T(r, f^{(k)}) \leqq \left(1 + \frac{1}{2k}\right)\left\{N\left(r, \frac{1}{f^{(k)} - a_1}\right) + N\left(r, \frac{1}{f^{(k)} - a_2}\right)\right\} + \varepsilon T(r, f^{(k)}) + S(r, f^{(k)}).$$

Thus

$$\begin{aligned} \left(1 + \frac{1}{2k}\right)\left\{\left(1 - \frac{N(r, 1/(f^{(k)} - a_1))}{T(r, f^{(k)})}\right) + \left(1 - \frac{N(r, 1/(f^{(k)} - a_2))}{T(r, f^{(k)})}\right)\right\} \\ < 1 + \frac{2}{2k} + \varepsilon + \frac{S(r, f^{(k)})}{T(r, f^{(k)})}, \end{aligned}$$

so that

$$\delta(a_1, f^{(k)}) + \delta(a_2, f^{(k)}) \leqq \frac{2k+2}{2k+1} + \varepsilon.$$

Since ε can be arbitrarily small, (2) is proved.

4. Proof of Theorem 1

If a_j ($j = 1, 2, \ldots, q$) are q distinct finite complex numbers, then we have

$$\sum_{j=1}^{q} m\left(r, \frac{1}{f^{(k)} - a_j}\right) \leqq m\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f^{(k)}). \tag{5}$$

Denote by e the union of exceptional sets corresponding to inequalities (1) and (5). Then e has finite linear measure.

We consider two cases which are mutually exclusive.

(1) $\limsup\limits_{\substack{r\to\infty \\ r\notin e}} \dfrac{\bar{N}(r, f)}{T(r, f^{(k)})} < \dfrac{1}{2k+1}$.

In this case, we have

$$\begin{aligned}\sum_{j=1}^{q} m\left(r, \frac{1}{f^{(k)} - a_j}\right) &\leqq T(r, f^{(k+1)}) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f^{(k)}) \\ &\leqq T(r, f^{(k)}) + \bar{N}(r, f) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f^{(k)}).\end{aligned} \tag{6}$$

Thus

$$\sum_{j=1}^{q} \delta(a_j, f^{(k)}) \leqq \limsup_{\substack{r\to\infty \\ r\notin e}} \frac{T(r, f^{(k)}) + \bar{N}(r, f)}{T(r, f^{(k)})} \leqq 1 + \frac{1}{2k+1}.$$

(2) $\limsup\limits_{\substack{r\to\infty \\ r\notin e}} \dfrac{\bar{N}(r, f)}{T(e, f^{(k)})} \geqq \dfrac{1}{2k+1}$.

Combining (1) and (6), we obtain

$$\begin{aligned}\sum_{j=1}^{q} m\left(r, \frac{1}{f^{(k)} - a_j}\right) &\leqq T(r, f^{(k)}) - k\bar{N}(r, f) + N(r, f) + \varepsilon T(r, f^{(k)}) + S(r, f^{(k)}) \\ &\leqq 2T(r, f^{(k)}) - 2k\bar{N}(r, f) + \varepsilon T(r, f^{(k)}) + S(r, f^{(k)}).\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{j=1}^{q} \delta(a_j, f^{(k)}) &\leqq \liminf_{\substack{r\to\infty \\ r\notin e}} \left\{2 - 2k\frac{\bar{N}(r, f)}{T(r, f^{(k)})}\right\} + \varepsilon + \limsup_{\substack{r\to\infty \\ r\notin e}} \frac{S(r, f^{(k)})}{T(r, f^{(k)})} \\ &\leqq \frac{2k+2}{2k+1} + \varepsilon.\end{aligned}$$

Since ε can be arbitrarily small, we also have

$$\sum_{j=1}^{q} \delta(a_j, f^{(k)}) \leqq \frac{2k+2}{2k+1}.$$

Because q can be arbitrarily large, the proof of Theorem 1 is complete.

5. Proof of Theorem 2 and corollaries

Similar to the case (2) of the Proof of Theorem 1, we have

$$m(r, f^{(k)}) + \sum_{j=1}^{q} m\left(r, \frac{1}{f^{(k)} - a_j}\right)$$
$$< 2T(r, f^{(k)}) - 2k\bar{N}(r, f) + \varepsilon T(r, f^{(k)}) + S(r, f^{(k)}).$$

Thus

$$\begin{aligned}
&\delta(\infty, f^{(k)}) + \sum_{j=1}^{q} \delta(a_j, f^{(k)}) \\
&\leqq \liminf_{r\to\infty} \left\{2 - \frac{2k\bar{N}(r, f)}{T(r, f^{(k)})} + \varepsilon + \frac{S(r, f^{(k)})}{T(r, f^{(k)})}\right\} \\
&\leqq \liminf_{r\to\infty} \left\{2 - \frac{2k\bar{N}(r, f)}{T(r, f^{(k)})}\right\} + \limsup_{r\to\infty} \left\{\varepsilon + \frac{S(r, f^{(k)})}{T(r, f^{(k)})}\right\} \qquad (7)\\
&\leqq 2 - 2k \limsup_{r\to\infty} \frac{\bar{N}(r, f)}{T(r, f^{(k)})} + \varepsilon.
\end{aligned}$$

Since

$$\frac{\bar{N}(r, f)}{T(r, f^{(k)})} \geqq \frac{\bar{N}(r, f)}{T(r, f) + k\bar{N}(r, f) + m(r, f^{(k)}/f)},$$

we have

$$\begin{aligned}
\limsup_{r\to\infty} \frac{\bar{N}(r, f)}{T(r, f^{(k)})} &\geqq \limsup_{r\to\infty} \frac{\bar{N}(r, f)}{T(r, f) + k\bar{N}(r, f) + m(r, f^{(k)}/f)} \\
&\geqq \frac{\limsup_{r\to\infty} \dfrac{\bar{N}(r, f)}{T(r, f)}}{\limsup_{r\to\infty} \left\{1 + k\dfrac{\bar{N}(r, f)}{T(r, f)} + \dfrac{m(r, f^{(k)}/f)}{T(r, f)}\right\}} \qquad (8)\\
&= \frac{1 - \Theta(\infty, f)}{1 + k(1 - \Theta(\infty, f))}.
\end{aligned}$$

Combining (7) and (8), let ε tend to zero and q tend to the infinity. We obtain finally

$$\sum_{a\in\hat{C}} \delta(a, f^{(k)}) \leqq 2 - \frac{2k(1-\Theta(\in, f))}{1+k(1-\Theta(\infty, f))}.$$

The following corollaries can be deduced from Theorem 2 immediately.

Corollary 1. *Suppose that $f(z)$ is transcendental meromorphic and of finite order in the finite plane. If $\Theta(\infty, f) < 1$, then we have*

$$\lim_{k\to\infty} \left\{ \sum_{a\in\hat{C}} \delta(a, f^{(k)}) \right\} = 0.$$

Corollary 2. *Let $f(z)$ be transcendental meromorphic and of finite order. If $\Theta(\infty, f) = 0$ (i.e. $\limsup_{r\to\infty}(\bar{N}(r, f)/T(r, f)) = 1$), then for any positive integer k, we have*

$$\sum_{a\in\hat{C}} \delta(a, f^{(k)}) \leqq \frac{2}{k+1}.$$

Corollary 3. *Let $f(z)$ be transcendental meromorphic and of finite order. If there exists a positive integer k_0 such that $\Sigma_{a\in\hat{C}}\delta(a, f^{(k_0)}) = 2$, then we have $\Theta(\infty, f) = 1$ (i.e. $\bar{N}(r, f) = o(T(r, f))$ as r tends to ∞).*

6. Problem of uniqueness

Using a similar idea, we are going to prove a theorem on the problem of uniqueness. In order to do it, we prove a preliminary lemma.

Lemma 2. *Let $f(z)$ be a transcendental meromorphic function in the plane and a_j $(j = 1, 2, \dots, q)$ be q $(\geqq 2)$ finite distinct complex values; then we have*

$$\left\{q - 1 - \frac{q-1}{kq+q-1}\right\} T(r, f^{(k)}) < \sum_{j=1}^{q} N\left(r, \frac{1}{f^{(k)} - a_j}\right) + \varepsilon T(r, f^{(k)}) + S(r, f^{(k)}), \tag{9}$$

where ε is any small positive number.

Proof. Applying the Nevanlinna Second Fundamental Theorem to $f(z)$ and $q+1$ complex values a_j $(j = 1, 2, \dots, q)$ and ∞, we have

$$(q-1)T(r,f^{(k)})<\bar{N}(r,f)+\sum_{j=1}^{q}N\left(r,\frac{1}{f^{(k)}-a_j}\right)$$

(10)

$$-N\left(r,\frac{1}{f^{(k+1)}}\right)+S(r,f^{(k)}).$$

Substituting (1) in (10) and noting

$$T(r,f^{(k)})\geqq N(r,f)+k\bar{N}(r,f),$$

we obtain

$$\bar{N}(r,f)<\frac{1}{kq+q-2}\sum_{j=1}^{q}N\left(r,\frac{1}{f^{(k)}-a_j}\right)+\varepsilon T(r,f^{(k)})+S(r,f^{(k)}).$$

Combining this inequality with (10), we deduce that

$$(q-1)T(r,f^{(k)})<\left(1+\frac{1}{kq+q-2}\right)\sum_{j=1}^{q}N\left(r,\frac{1}{f^{(k)}-a_j}\right)$$

$$-N\left(r,\frac{1}{f^{(k+1)}}\right)+\varepsilon T(r,f^{(k)})+S(r,f^{(k)}).$$

Dividing every term by $1+1/(kq+q-2)$, we obtain the inequality (9).

Now, suppose that $f_1(z)$ and $f_2(z)$ are two transcendental meromorphic functions in the finite plane. Let a_j $(j=1,2,\ldots,q)$ be q $(\geqq 4)$ distinct finite complex values and k be a positive integer. We denote by $N_{1,2}^{(k)}(r,a_j)$ $(j=1,2,\ldots,q)$ the counting function with respect to all the non-common zeros of $f_1^{(k)}(z)-a_j$ and $f_2^{(k)}(z)-a_j$ in $|z|\leqq r$. Multiple zeros should be counted with their multiplicities. Under these notations, we have

Theorem 3. *If*

$$\sum_{j=1}^{q}\left\{1-\limsup_{r\to\infty}\frac{N_{1,2}^{(k)}(r,a_j)}{T(r,f_1^{(k)})+T(r,f_2^{(k)})}\right\}>3+\frac{q-1}{kq+q-1},\tag{11}$$

then we have $f_1^{(k)}(z)\equiv f_2^{(k)}(z)$. *Therefore* $f_1(z)\equiv f_2(z)+P_{k-1}(z)$, *where* $P_{k-1}(z)$ *is a polynomial of degree* $k-1$.

In fact, if $f_1^{(k)}$ is not identical to $f_2^{(k)}$, we apply Lemma 2 to f_1 and f_2 respectively and have

$$\left\{q-1-\frac{q-1}{kq+q-1}\right\}T(r, f_l^{(k)})$$
$$< \sum_{j=1}^{q} N\left(r, \frac{1}{f_l^{(k)}-a_j}\right)+\varepsilon T(r, f_l^{(k)})+S(r, f_l^{(k)}) \quad (l=1,2).$$

Then we add these two inequalities and note

$$\sum_{j=1}^{q}\left\{N\left(r, \frac{1}{f_1^{(k)}-a_j}\right)+N\left(r, \frac{1}{f_2^{(k)}-a_j}\right)\right\}=\sum_{j=1}^{q}\{N_{1,2}^{(k)}(r, a_j)+2N_0^{(k)}(r, a_j)\},$$

where $N_0^{(k)}(r, a_j)$ $(j=1, 2, \ldots, q)$ denotes the counting function with respect to all the common zeros of $f_1^{(k)}(z)-a_j$ and $f_2^{(k)}(z)-a_j$ in $|z| \leqq r$. Since $f_1^{(k)}$ is not identical to $f_2^{(k)}$, every common zero of $f_1^{(k)}-a_j$ and $f_2^{(k)}-a_j$ must be a pole of $1/(f_1^{(k)}-f_2^{(k)})$, so that

$$\sum_{j=1}^{q} N_0^{(k)}(r, a_j) \leqq N\left(r, \frac{1}{f_1^{(k)}-f_2^{(k)}}\right)$$
$$\leqq T(r, f_1^{(k)})+T(r, f_2^{(k)})+O(1).$$

Thus

$$\left\{q-3-\frac{q-1}{kq+q-1}\right\}(T(r, f_1^{(k)})+T(r, f_2^{(k)}))$$
$$< \sum_{j=1}^{q} N_{1,2}^{(k)}(r, a_j)+\varepsilon(T(r, f_1^{(k)})+T(r, f_2^{(k)}))+O\{\log(rT(r, f_1^{(k)})T(r, f_2^{(k)}))\},$$

except for r in a set with linear measure zero. This inequality yields that

$$\sum_{j=1}^{q}\left\{1-\limsup_{r\to\infty}\frac{N_{1,2}^{(k)}(r, a_j)}{T(r, f_1^{(k)})+T(r, f_2^{(k)})}\right\} \leqq 3+\frac{q-1}{kq+q-1},$$

which contradicts (11). Therefore the proof of Theorem 3 is complete.

Remark. An interesting problem is whether the bounds of Theorem 1 and Theorem 2 are sharp or not.

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