DEFICIENT VALUES AND ANGULAR DISTRIBUTION
OF ENTIRE FUNCTIONS

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ABSTRACT. Let $f(z)$ be an entire function of positive and finite order μ. If $f(z)$ has a finite number of Borel directions of order $\geq \mu$, then the sum of numbers of finite nonzero deficient values of $f(z)$ and all its primitives does not exceed 2μ. The proof is based on several lemmas and application of harmonic measure.

1. Introduction. Let $f(z)$ be an arbitrary entire function of finite and positive order λ. A. Denjoy conjectured that $f(z)$ has at most 2λ distinct asymptotic values. L. Ahlfors succeeded in proving this conjecture later. He proved further that any entire function of finite lower order μ has 2μ asymptotic values at most [6, pp. 303–307].

Since there are some similarities between asymptotic values and deficient values of entire functions, one may expect that any entire function of finite order would have a finite number of deficient values. N. U. Arakelyan [1], however, constructed for the first time an entire function of order one having infinite deficient values.

It is natural to ask: for an entire or meromorphic function of finite order λ or finite lower order μ, under what conditions does the number of its deficient values have an upper bound depending only on λ or μ? We ask for conditions which ensure that the number of deficient values of an entire function satisfy the bound of Ahlfors's theorem.

One such condition is that entire or meromorphic functions have deficiency sum two. This is actually a part of F. Nevanlinna's conjecture. In the case of entire functions, cf. A. Pfluger [7], A. Edrei and W. Fuchs [4] proved several interesting theorems. Then A. Weitsman [10] obtained partial results for meromorphic functions and D. Drasin [2, 3] recently made a great contribution to prove F. Nevanlinna's conjecture completely. For entire functions, we also posed a similar condition [13]

$$\sum_{j=-\infty}^{+\infty} \sum_{a \neq 0, \infty} \delta(a, f^{(j)}) = 1$$

and obtained the corresponding result.

In this paper, another condition will be introduced.

Let $f(z)$ be an entire function of lower order μ, where $0 < \mu < \infty$. A ray $\arg z = \theta_0$ $(0 \leq \theta_0 < 2\pi)$ is named a *Borel direction of order $\geq \mu$ of* $f(z)$, if for any

positive number ε, the inequality

$$\varlimsup_{r \to +\infty} \frac{\log n(r, \theta_0 - \varepsilon, \theta_0 + \varepsilon, f = a)}{\log r} \geq \mu$$

holds for any finite complex value a, with possibly one exceptional value, where $n(r, \theta_0 - \varepsilon, \theta_0 + \varepsilon, f = a)$ denotes the number of zeros of $f(z) - a$ in the region $(|z| \leq r) \cap (\theta_0 - \varepsilon \leq \arg z \leq \theta_0 + \varepsilon)$, multiple zeros being counted with their multiplicities.

The aim of this paper is to prove the following theorem.

THEOREM. *Let $f(z)$ be an entire function of lower order μ, where $0 < \mu < +\infty$. If $q < \infty$ is the number of Borel directions of order $\geq \mu$ of $f(z)$ and p_l $(l = 0, -1, -2, \dots)$ denotes the number of finite nonzero deficient values of $f^{(l)}(z)$ $(l = 0, -1, -2, \dots; f^{(0)} \equiv f(z))$,[1] then we have $\sum_{l=0}^{-\infty} p_l \leq 2\mu$.*

2. A property of monotonic functions.
2.1. We first establish a lemma on monotonic functions.

LEMMA 1. *Let $f(z)$ be an entire function of finite lower order μ, where $0 < \mu < +\infty$. If β_1 and β_2 are two arbitrary numbers with $1 < \beta_1 < \beta_2$, then there exists a sequence $\{t_k\}$ of positive numbers tending to infinity such that*

$$(2.1) \qquad \lim_{k \to +\infty} \frac{\log T(t_k, f)}{\log t_k} = \mu$$

and the inequalities

$$(2.2) \qquad T(\beta_1 t_k, f) < (\beta_1^\mu + o(1)) T(t_k, f)$$

and

$$(2.3) \qquad T(\beta_2 t_k, f) < (\beta_1^\mu \beta_2^\mu + o(1)) T(t_k, f)$$

hold, when k tends to infinity.

PROOF. Since the lower order of $f(z)$ equals μ, there is a sequence $\{r_k\}$ $(k = 2, 3, \dots)$ of positive numbers tending to infinity such that

$$(2.4) \qquad (r_k^{1-1/k})^{\mu - 1/k^2} < T(r_k^{1-1/k}, f) \leq T(r_k, f) < r_k^{\mu + 1/k^2}$$

and

$$(2.5) \qquad \log r_k > k^3(\mu + 2/k) \log(\beta_1 \beta_2).$$

We shall prove the existence of a corresponding sequence $\{t_k\}$ which satisfies

$$(2.6) \qquad t_k \in [r_k^{1-1/k}, r_k], \qquad k = 1, 2, 3, \dots,$$

$$(2.7) \qquad T(\beta_1 t_k, f) < \beta_1^{\mu + 2/k} T(t_k, f)$$

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[1]$f^{(l)}(z)$ $(l < 0)$ is the primitive of order $|l|$ of $f(z)$.

and

(2.8) $$T(\beta_2 t_k, f) < (\beta_1\beta_2)^{\mu+2/k}T(t_k, f),$$

hence the conclusion of lemma.

Let us now prove the existence of $\{t_k\}$ satisfying (2.6), (2.7) and (2.8). Denote

$$E = \{t: r_k^{1-1/k} \le t \le r_k, T(\beta_2 t, f) \ge (\beta_1\beta_2)^{\mu+2/k}T(t, f)\},$$
$$\tau_1 = \min E, \quad \tau_j = \min\{E \cap [\beta_2\tau_{j-1}, r_k]\}, \qquad j = 2, 3, \ldots, J+1,$$

where J is chosen so that $\tau_{J+1} < r_k \le \beta_2\tau_{J+1}$. For every j $(0 \le j \le J)$ set further

$$e_j = \{t: \beta_1\tau_j \le t \le \tau_{j+1}, T(\beta_1 t, f) \ge \beta_1^{\mu+2/k}T(t, f)\},$$
$$\sigma_{j,1} = \min e_j,$$
$$\sigma_{j,l} = \min\{e_j \cap [\beta_1\sigma_{j,l-1}, \tau_{j+1}]\}, \qquad l = 2, 3, \ldots, L_j+1,$$

where $\tau_0 = r_k^{1-1/k}$ and $\sigma_{j,L_j+1} < \tau_{j+1} \le \beta_1\sigma_{j,L_j+1}$.

By the above notations and (2.4), we have

$$
\begin{aligned}
(r_k^{1-1/k})^{\mu-1/k^2} &< T(r_k^{1-1/k}, f) \le T(\sigma_{0,1}, f) \\
&\le \beta_1^{-(\mu+2/k)}T(\sigma_{0,2}, f) \le \cdots \le \beta_1^{-L_0(\mu+2/k)}T(\sigma_{0,L_0+1}, f) \\
&\le \beta_1^{-L_0(\mu+2/k)}T(\tau_1, f) \le \beta_1^{-L_0(\mu+2/k)}(\beta_1\beta_2)^{-(\mu+2/k)}T(\beta_2\tau_1, f) \\
&\le \cdots \le \beta_1^{-(L_0+L_1)(\mu+2/k)}(\beta_1\beta_2)^{-2(\mu+2/k)}T(\beta_2\tau_2, f) \\
&\le \cdots \le \beta_1^{-(\mu+2/k)\sum_{j=0}^{J}L_j}(\beta_1\beta_2)^{-J(\mu+2/k)}T(r_k, f) \\
&< \beta_1^{-(\mu+2/k)\sum_{j=0}^{J}L_j}(\beta_1\beta_2)^{-J(\mu+2/k)}r_k^{\mu+1/k^2},
\end{aligned}
$$

so that

$$\left(\mu + \frac{2}{k}\right)\sum_{j=0}^{J}L_j\log\beta_1 + J\left(\mu + \frac{2}{k}\right)\log(\beta_1\beta_2) < \left(\mu + \frac{2}{k} - \frac{1}{k^2}\right)\frac{\log r_k}{k}.$$

Thus

$$\sum_{j=0}^{J}L_j\log\beta_1 + (J+1)\log\beta_1 + (J+1)\log\beta_2$$
$$< \left\{1 - \frac{1}{k^2\left(\mu+\frac{2}{k}\right)} + \frac{k\log(\beta_1\beta_2)}{\log r_k}\right\}\frac{\log r_k}{k} < \frac{\log r_k}{k}$$

by (2.5). Consequently

$$\int_E \frac{dt}{t} + \sum_{j=0}^{J}\int_{e_j}\frac{dt}{t} < \int_{r_k^{1-1/k}}^{r_k}\frac{dt}{t},$$

i.e. there is a value t_k not in the set $E \cup (\bigcup_{j=0}^{J}e_j)$.

3. Several known results.

3.1. In the sequel, we need several known results.

LEMMA A [12]. *Let $f(z)$ be an entire function of lower order μ, where $0 < \mu < +\infty$. If $f(z)$ has no Borel direction of order $\geq \mu$ in the angle $\theta_1 < \arg z < \theta_2$, then there exist two finite distinct complex values a_j $(j = 1, 2)$ such that for any number ε, $0 < \varepsilon < (\theta_2 - \theta_1)/2$, we have*

$$(3.1) \qquad \overline{\lim_{r \to +\infty}} \frac{\log\{\sum_{j=1}^{2} n(r, \theta_1 + \varepsilon, \theta_2 - \varepsilon, f = a_j)\}}{\log r} < \mu.$$

LEMMA B [12]. *Let $f(z)$ be an entire function of lower order μ, where $0 < \mu < +\infty$. If the number of its Borel directions of order $\geq \mu$ is finite and the minimum angular separation of the Borel directions is equal to ω, then the order λ of $f(z)$ must be finite and $\mu \leq \lambda \leq \pi/\omega$.*

LEMMA C [13, p. 468]. *Let $g(\varsigma)$ be meromorphic on $|\varsigma| \leq R$ $(< +\infty)$. If*

$$\mathfrak{N} = n(R, g = 0) + n(R, g = 1) + n(R, g = \infty)$$

and d (> 0) is the minimum distance from the origin to all the points of $g(\varsigma) = 0, 1$ or ∞, then

$$(3.2) \qquad \begin{aligned} T(r, g) &< \frac{CR\mathfrak{N}}{R - r}\left\{\log\frac{R\mathfrak{N}}{d} + \log^+ R + \log^+ \frac{1}{R} + \log\frac{2R}{R - r}\right\} \\ &\quad + \log^+ |g(0)| \end{aligned}$$

for $0 \lesssim r < R$, where C is a numerical constant.

4. Property of functions having deficient values.

4.1. We need more preparations to prove the main result formulated in the introduction.

LEMMA 2. *Let $f(z)$ be an entire function of lower order μ, where $0 < \mu < +\infty$. If $f(z)$ has p_0 $(0 \leq p_0 < +\infty)$ finite deficient values a_j $(j = 1, 2, \ldots, p_0)$, $f^{(-1)}(z)$ has p_{-1} $(0 \leq p_{-1} < +\infty)$ finite deficient values b_l $(l = 1, 2, \ldots, p_{-1})$ and*

$$\delta = \min_{\substack{1 \leq j \leq p_0 \\ 1 \leq l \leq p_{-1}}} \{\delta(a_j, f), \delta(b_l, f^{(-1)})\},$$

then there exists a sequence $\{R_k\}$ of positive numbers such that

$$(4.1) \qquad\qquad t_k \leq R_k \leq 2t_k,$$

$$(4.2) \qquad \operatorname{mes} E_{kj} = \operatorname{mes} E\{\theta\colon \log|f(R_k e^{i\theta}) - a_j| \leq -(\delta/4)T(R_k, f)\} > B$$
$$(j = 1, 2, \ldots, p_0)$$

and

$$(4.3) \quad \operatorname{mes} E_{kl}^{(-1)} = \operatorname{mes} E\{\theta\colon \log|f^{(-1)}(R_k e^{i\theta}) - b_l| \leq -(\delta/4)T(R_k, f)\} > B$$
$$(l = 1, 2, \ldots, p_{-1})$$

where k is sufficiently large. The sequence $\{t_k\}$ is determined by Lemma 1 with $\beta_1 = 8$ and $\beta_2 = 4 \cdot 10^{4V}$ (V will be given by (6.6)) and

$$(4.4) \qquad B = B(\delta, \mu, p_0, p_{-1}) = \frac{\delta\pi}{6 \cdot 8^\mu (6 + \log\{40(p_0 + p_{-1})e\}/\log\frac{4}{3})} > 0.$$

PROOF. According to Lemma 1, there is a sequence $\{t_k\}$ satisfying

$$(4.5) \qquad \lim_{k \to +\infty} \frac{\log T(t_k, f)}{\log t_k} = \mu,$$

$$(4.6) \qquad T(8t_k, f) < (8^\mu + o(1))T(t_k, f)$$

and

$$T(4 \cdot 10^{4V} t_k, f) < \{(8 \cdot 4 \cdot 10^{4V})^\mu + o(1)\}T(t_k, f).$$

Let k be sufficiently large and fixed. In $|z| \leq 3t_k$, denote by α_{js} $(s = 1, 2, \ldots, n_j;$ $n_j = n(3t_k, f = a_j))$ the zeros of $f(z) - a_j$ and by β_{lu} $(u = 1, 2, \ldots, n_l^{(-1)}; n_l^{(-1)} = n(3t_k, f^{(-1)} = b_l))$ the zeros of $f^{(-1)}(z) - b_l$. From the Boutroux-Cartan theorem [9], we have

$$(4.7) \qquad \prod_{s=1}^{n_j} |z - \alpha_{js}| > H^{n_j} \qquad (j = 1, 2, \ldots, p_0)$$

and

$$(4.8) \qquad \prod_{u=1}^{n_l^{(-1)}} |z - \beta_{lu}| > H^{n_l^{(-1)}} \qquad (l = 1, 2, \ldots, p_{-1})$$

except a set which can be enclosed in a finite number of disks with the sum of total radii not exceeding $2e(p_0 + p_{-1})H$. The union of these disks is denoted by (γ).

Choose

$$(4.9) \qquad H = t_k/8e(p_0 + p_{-1}).$$

There exists a positive number R_k such that

$$R_k \in [t_k, 2t_k], \quad (|z| = R_k) \cap (\gamma) = \varnothing.$$

Since $R_k e^{i\theta} \notin (\gamma)$ $(0 \leq \theta < 2\pi)$, (4.7), (4.9) and (4.6), we obtain

(4.10)

$$
\begin{aligned}
\log \frac{1}{|f(R_k e^{i\theta}) - a_j|} &\leq \frac{3t_k + R_k}{3t_k - R_k} m\left(3t_k, \frac{1}{f - a_j}\right) \\
&\quad + \sum_{s=1}^{n_j} \log\left|\frac{(3t_k)^2 - \overline{\alpha}_{js} R_k e^{i\theta}}{3t_k(R_k e^{i\theta} - \alpha_{js})}\right| \\
&\leq 5m\left(3t_k, \frac{1}{f - a_j}\right) + n_j\left(\log 5t_k + \log\frac{1}{H}\right) \\
&\leq 5m\left(3t_k, \frac{1}{f - a_j}\right) + \frac{N(4t_k, f = a_j)}{\log\frac{4}{3}}\log\{40(p_0 + p_{-1})e\} \\
&< \left\{5 + \frac{\log\{40(p_0 + p_{-1})e\}}{\log\frac{4}{3}} + o(1)\right\} T(8t_k, f) \\
&< 8^\mu\left\{5 + \frac{\log\{40(p_0 + p_{-1})e\}}{\log\frac{4}{3}} + o(1)\right\} T(t_k, f).
\end{aligned}
$$

On the other hand, we have

$$\frac{\delta}{2} T(R_k, f) < \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(R_k e^{i\theta}) - a_j|} \, d\theta$$

$$< \frac{1}{2\pi} \int_{E_{kj}} \log^+ \frac{1}{|f(R_k e^{i\theta}) - a_j|} \, d\theta + \frac{\delta}{4} T(R_k, f).$$

Substituting (4.10) into this inequality, we obtain

$$\frac{\delta}{4} T(R_k, f) < \frac{1}{2\pi} \int_{E_{kj}} \log^+ \frac{1}{|f(R_k e^{i\theta}) - a_j|} \, d\theta$$

$$< \frac{1}{2\pi} 8^\mu \left\{ 5 + \frac{\log\{40(p_0 + p_{-1})e\}}{\log \frac{4}{3}} + o(1) \right\} T(R_k, f) \operatorname{mes} E_{kj}.$$

Therefore (4.2) is proved.

Similarly we obtain

$$\log \frac{1}{|f^{(-1)}(R_k e^{i\theta}) - b|} < \left\{ 5 + \frac{\log\{40(p_0 + p_{-1})e\}}{\log \frac{4}{3}} + o(1) \right\} T(4t_k, f^{(-1)}).$$

For sufficiently large r, from

$$\begin{aligned}
(4.11) \qquad T(r, f^{(-1)}) &\leq \log M(r, f^{(-1)}) \\
&\leq \log\{|f^{(-1)}(0)| + r M(r, f)\} \\
&< (3 + o(1)) T(2r, f)
\end{aligned}$$

and (4.6), we have

$$T(4t_k, f^{(-1)}) < (3 + o(1)) T(8t_k, f) < (3 \cdot 8^\mu + o(1)) T(t_k, f).$$

Hence

$$\log \frac{1}{|f^{(-1)}(R_k e^{i\theta}) - b_l|} < 3 \cdot 8^\mu \left\{ 5 + \frac{\log\{40(p_0 + p_{-1})e\}}{\log \frac{4}{3}} + o(1) \right\} T(t_k, f).$$

The same procedure yields

$$(4.12) \quad \frac{\delta}{4} T(R_k, f^{(-1)})$$

$$< \frac{1}{2\pi} 3 \cdot 8^\mu \left\{ 5 + \frac{\log\{40(p_0 + p_{-1})e\}}{\log \frac{4}{3}} + o(1) \right\} T(R_k, f) \operatorname{mes} E_{kl}^{(-1)}.$$

Since

$$T(R_k, f) \leq T(R_k, f^{(-1)}) + m(R_k, f/f^{(-1)})$$

and

$$m\left(R_k, \frac{f}{f^{(-1)}}\right) < O(1) + 4\log^+ \rho + 3\log^+ \frac{1}{\rho - R_k} + 4\log^+ T(\rho, f^{(-1)})$$

$$(R_k < \rho)$$

we have

$$\begin{aligned}
\log^+ T(\rho, f^{(-1)}) &< \log^+\{4T(8t_k, f)\} \\
&< \log^+\{4(8^\mu + o(1)) T(t_k, f)\} = O(\log R_k)
\end{aligned}$$

by choosing $\rho = 4t_k$, (4.11), (4.6) and (4.5). Thus

(4.13) $$(\delta/4 - o(1))T(R_k, f) < (\delta/4)T(R_k, f^{(-1)}).$$

Comparing (4.12) with (4.13), we derive (4.3).

REMARK. We may replace $f^{(-1)}(z)$ by $f^{(l)}(z)$ $(-1 \geq l \geq -L)$ and obtain a similar result.

5. Properties of functions having no Borel directions in an angle.

5.1. Now we shall establish a property of meromorphic functions without Borel directions in an angle.

LEMMA 3. *Let $f(z)$ be an entire function of lower order μ and order λ, where $0 < \mu \leq \lambda < +\infty$. Suppose that $f(z)$ has no Borel direction of order $\geq \mu$ in the angle G: $\theta_1 < \arg z < \theta_2$. If there are positive numbers δ and B, a finite complex value a_0 and a positive and sufficiently large number[2] R_k such that*

(5.1) $$\operatorname{mes} E\{\theta: \theta_1 < \theta < \theta_2, \log|f(R_k e^{i\theta}) - a_0| \leq -(\delta/4)T(R_k, f)\} > B,$$

then for two positive numbers α and Q, $0 < \alpha < \frac{1}{4}(\theta_2 - \theta_1)$, $Q > 1$, we have

(5.2) $$\log|f(z) - a_0| < -C_0 T(R_k, f)$$

and

(5.3) $$\log|f^{(j)}(z)| < -C_0 T(R_k, f) + \log\left\{ j! \left(\frac{20 \cdot 10^{4Q}}{\alpha R_k} \right)^j \right\} \qquad (j = 1, 2, \dots)$$

in the region

(5.4) $$D_k: (10^{-4Q} R_k \leq |z| \leq 10^{4Q} R_k) \cap (\theta_1 + \alpha \leq \arg z \leq \theta_2 - \alpha),$$

where

(5.5)
$$C_0 = \frac{\delta}{4\{4(5 + 4\log\frac{1}{h})\}^{2N_1 + N_2}},$$
$$N_1 = \left[\frac{10\pi}{\alpha} \right], \quad N_2 = \left[\frac{20(10^{4Q} - 1)}{\alpha} \right] + 4,$$

and

(5.6) $$h = \frac{B\alpha}{8(2e+1)(10\pi + \alpha)}.$$

PROOF. Choose positive number $\alpha < \min(B/4, (\theta_2 - \theta_1)/2)$. It is easy to see that

$$\operatorname{mes}\left\{ \varphi: \theta_1 + \frac{4}{5}\alpha \leq \varphi \leq \theta_2 - \frac{4}{5}\alpha, \log|f(R_k e^{i\varphi}) - a_0| \leq -\frac{\delta}{4}T(R_k, f) \right\} > \frac{B}{2}.$$

We divide G into N_1 equal angles G_ν $(\nu = 1, 2, \dots, N_1; N_1 = [10\pi/\alpha] + 1)$, so the opening of G_ν does not exceed $\alpha/5$. There is an angle G_{ν_0} such that

$$\operatorname{mes}\left\{ \varphi: R_k e^{i\varphi} \in G_{\nu_0}, \log|f(R_k e^{i\varphi}) - a_0| \leq -\frac{\delta}{4}T(R_k, f) \right\}$$
$$> \frac{B}{2N_1} \geq \frac{\alpha B}{2(10\pi + \alpha)}.$$

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[2]In the late use of §7, there is a sequence of positive numbers which tend to infinity and satisfy (5.1) for every k. For the abbreviation of notations, we consider only one sufficiently large number R_k here.

Since $f(z)$ has no Borel direction of order $\geq \mu$ in G, two finite distinct complex values β_l ($l = 1, 2$) and a positive number $\tau < \mu$ can be found with

$$\sum_{l=1}^{2} n\left\{r, \theta_1 + \frac{\alpha}{10}, \theta_2 - \frac{\alpha}{10}, f = \beta_l\right\} < r^\tau$$

by Lemma A, provided that r is sufficiently large.

Write

$$n_1 = \sum_{l=1}^{2} n\left(\left(1 + \frac{3}{5}\alpha\right) R_k, \theta_1 + \frac{\alpha}{10}, \theta_2 - \frac{\alpha}{10}, f = \beta_l\right)$$
$$+ n\left(\left(1 + \frac{3}{5}\alpha\right) R_k, \theta_1 + \frac{\alpha}{10}, \theta_2 - \frac{\alpha}{10}, f = a_0\right).$$

Then we construct some disks, having their centres at every zero of $f(z) - \beta_l$ ($l = 1, 2$) and of $f(z) - a_0$ in the region $(|z| \leq (1 + \frac{3}{5}\alpha)R_k) \cap (\theta_1 + \alpha/10 \leq \arg z \leq \theta_2 - \alpha/10)$ and having the same radius $\alpha h R_k / 2(n_1 + 1)$, where h is given by (5.6). The union of these disks is denoted by $(\gamma)_1$.

Let z_1 be the point of intersection of $|z| = R_k$ and the bisector of G_{ν_0} and let

(5.7) $\Gamma_k : |z - z_1| < \alpha R_k / 10.$

Denote by d_s ($s = 1, 2, \ldots, n_2$) the zeros of $f(z) - a_0$ in $|z - z_2| < \frac{3}{10}\alpha R_k$, where z_2 is an arbitrary point in $\Gamma_k \setminus (\gamma)_1$. According to the Boutroux-Cartan theorem, we have

$$\prod_{s=1}^{n_2} |z - d_s| > \left(\frac{h\alpha R_k}{2}\right)^{n_2},$$

except a set contained in several disks with the sum of radii not exceeding $eh\alpha R_k$. Denote by $(\gamma)_2$ the union of these disks.

Since the total sum of diameters of $(\gamma)_1$ and $(\gamma)_2$ does not exceed

$$\alpha h R_k + 2eh\alpha R_k = \frac{\alpha^2 K}{8(10\pi + \alpha)} R_k,$$

there is a point z_3 in $\Gamma_k \setminus ((\gamma)_1 \cup (\gamma)_2)$ such that

(5.8) $\log|f(z_3) - a_0| < -(\delta/4) T(R_k, f).$

Applying the Poisson-Jensen's formula [5], we have

$$\begin{aligned}
\text{(5.9)} \quad \log \frac{1}{|f(z_3) - a_0|} &\leq 5m\left(\frac{3}{10}\alpha R_k, z_2, \frac{1}{f - a_0}\right) + \sum \log \left|\frac{(\frac{3}{10}\alpha R_k)^2 - \overline{d}_s z_3}{\frac{3}{10}\alpha R_k (z_3 - d_s)}\right| \\
&< \left(5 + 4\log\frac{1}{h}\right) T\left(\frac{2}{5}\alpha R_k, z_2, \frac{1}{f - a_0}\right),
\end{aligned}$$

where

$$m\left(\frac{3}{10}\alpha R_k, z_2, 1/(f - a_0)\right) \quad \text{and} \quad T\left(\frac{2}{5}\alpha R_k, z_2, 1/(f - a_0)\right)$$

denote

$$m\left(\frac{3}{10}\alpha R_k, 1/(f(z + z_2) - a_0)\right) \quad \text{and} \quad T\left(\frac{2}{5}\alpha R_k, 1/(f(z + z_2) - a_0)\right)$$

respectively. Further Lemma C gives

(5.10)
$$T\left(\frac{2}{5}\alpha R_k, z_2, \frac{1}{f-a_0}\right)$$
$$\leq T\left(\frac{2}{5}\alpha R_k, z_2, \frac{1/(f-a_0)}{1/(f-a_0)-1/(\beta_2-a_0)} \cdot \frac{1/(\beta_1-a_0)-1/(\beta_2-a_0)}{1/(\beta_1-a_0)}\right)$$
$$+ \log\left|\frac{1}{f(z_2)-a_0} - \frac{1}{\beta_2-a_0}\right| + C$$
$$< C\left(\sum_{l=1}^{2} n\left(\frac{1}{2}\alpha R_k, z_2, f=\beta_l\right)\right)$$
$$\times \left\{\log\frac{\sum_{l=1}^{2} n(\frac{1}{2}\alpha R_k, z_2, f=\beta_l)+1}{h/(n_1+1)} + \log R_k + \log 10\right\}$$
$$+ 2\log^+\frac{1}{|f(z_2)-a_0|} + C,$$

where C is a numerical constant which need not be the same at each occurrence.

In view of

$$\sum_{l=1}^{2} n\left(\frac{1}{2}\alpha R_k, z_2, f=\beta_l\right) = O(R_k^\tau)$$

and $n_1 < R_k^{\lambda+1}$, we obtain from (5.8), (5.9) and (5.10) that

$$\frac{\delta}{8}T(R_k, f) < 2\left(5 + 4\log\frac{1}{h}\right)\log\frac{1}{|f(z_2)-a_0|},$$

i.e.

$$\log\frac{1}{|f(z_2)-a_0|} > \frac{\delta}{16(5+4\log\frac{1}{h})}T(R_k, f).$$

We rotate Γ_k about the origin $\alpha/10$ in succession. Then a chain Γ_Σ of $2N_1$ disks at most can be obtained to cover the region

$$\left(R_k - \frac{\sqrt{3}}{20}\alpha R_k \leq |z| \leq R_k + \frac{\sqrt{3}}{20}\alpha R_k\right) \cap \left(\theta_1 + \frac{4}{5}\alpha \leq \arg z \leq \theta_2 - \frac{4}{5}\alpha\right).$$

By the same procedure, we can prove that

(5.11)
$$|f(z)-a_0| < \exp\left(-\frac{\delta}{4\{4(5+4\log\frac{1}{h})\}^{2N_1}}T(R_k, f)\right)$$

for any point $z \in (\Gamma_\Sigma \setminus (\gamma)_1)$. Since the total sum of diameters of disks in $(\gamma)_1$ does not exceed $\alpha h R_k < \alpha R_k/80$, (5.11) holds without exception in the region

$$(R_k - \tfrac{1}{20}\alpha R_k \leq |z| \leq R_k + \tfrac{1}{20}\alpha R_k) \cap (\theta_1 + \tfrac{33}{40}\alpha \leq \arg z \leq \theta_2 - \tfrac{33}{40}\alpha)$$

by the maximum modulus theorem.

Let $z_0 = R_k e^{i\theta_0}$ be an arbitrary point on

$$(|z| = R_k) \cap (\theta_1 + \tfrac{33}{40}\alpha \leq \arg z \leq \theta_2 - \tfrac{33}{40}\alpha).$$

We make two concentric disks

$$\Gamma'_k \colon |z - z_0| \leq \frac{1}{20}\alpha R_k, \quad \Gamma''_k \colon |z - z_0| \leq \frac{\alpha R_k}{20 \cdot 10^{4Q}}$$

where $Q > 1$ is mentioned in the lemma. With the centre of Γ'_k on the segment

$$L_1 \colon \left\{ re^{i\theta_0} \colon R_k \leq r \leq \left(10^{4Q} + \frac{\alpha}{10}\right) R_k \right\},$$

we move Γ'_k along L_1 by distance $\alpha R_k/20$ each time. Then L_1 is covered by $N_2 = [(20 \cdot 10^{4Q})/\alpha] + 4$ at most of these disks. Similarly, we move Γ''_k by distance $\alpha R_k/(20 \cdot 10^{4Q})$ each time along

$$L_2 \colon \{(1/10^{4Q} - \alpha/(5 \cdot 10^{4Q}))R_k \leq r \leq R_k\}$$

and L_2 is covered by at most N_2 of these disks.

Therefore we have (5.2) in the region

$$\left\{ \left(\frac{1}{10^{4Q}} - \frac{\alpha}{5 \cdot 10^{4Q}}\right) R_k \leq |z| \leq \left(10^{4Q} + \frac{\alpha}{10}\right) R_k \right\}$$

$$\cap \left(\theta_1 + \frac{9}{10}\alpha \leq \arg z \leq \theta_2 - \frac{9}{10}\alpha\right),$$

hence (5.3) by the Cauchy inequality.

5.2. We establish a corresponding lemma for the primitive.

LEMMA 4. *Let $f(z)$ be an entire function of lower order μ and order λ, where $0 < \mu \leq \lambda < +\infty$. Suppose that $f(z)$ has no Borel direction of order $\geq \mu$ in the angle $\theta_1 < \arg z < \theta_2$. If there are two positive numbers δ and B, a finite complex value b_0 and a sufficiently large positive number R_k such that*

$$(5.12) \quad \operatorname{mes} E_k^{(-1)} = \operatorname{mes} E\{\theta \colon \theta_1 < \theta < \theta_2, \log|f^{(-1)}(R_k e^{i\theta}) - b_0| $$
$$\leq -(\delta/4)T(R_k, f)\} > B$$

then we have in D_k defined by (5.4)

$$(5.13) \qquad \log|f^{(-1)}(z) - b_0| < -(\delta/8)T(R_k, f) < -C_0 T(R_k, f),$$

$$(5.14) \qquad\qquad \log|f(z)| < -(C_0/2)T(R_k, f)$$

and

$$(5.15) \quad \log|f^{(j)}(z)| < -\frac{C_0}{2}T(R_k, f) + \log\left\{ j! \left(\frac{20 \cdot 10^{4Q}}{\alpha R_k}\right)^j \right\}, \qquad j = 1, 2, \ldots,$$

where C_0 is given by (5.5).

PROOF. Write

$$N = n(2R_k, f^{(-1)} = b_0) + n(2R_k, f = 0)$$

and $d = BR_k/4N$. Let (γ) be the union of disks, having centres at zeros of $f^{(-1)}(z) - b_0$ and of $f(z)$ in $|z| \leq 2R_k$ and having the same radius d. Since the total sum of radii does not exceed $Nd = BR_k/4$, we have

$$(5.16) \qquad \operatorname{mes} E_k^{(-1)*} = \operatorname{mes} E\{\theta \colon \theta \in E_k^{(-1)}, R_k e^{i\theta} \bar{\varepsilon}(\gamma)\} > B/2.$$

When $\theta \in E_k^{(-1)^*}$, differentiation of the Poisson-Jensen formula gives

$$\left| \frac{f(R_k e^{i\theta})}{f^{(-1)}(R_k e^{i\theta}) - b_0} \right| \le \frac{4}{R_k} \cdot \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f^{(-1)}(2R_k e^{i\varphi}) - b_0 \right| d\varphi$$
$$+ \sum \frac{(2R_k)^2}{2R_k^2 |R_k e^{i\theta} - \beta_\nu|},$$

where β_ν are zeros of $f^{(-1)}(z) - b_0$ in $|z| \le 2R_k$. Thus

$$\log |f(R_k e^{i\theta})| \le \log |f^{(-1)}(R_k e^{i\theta}) - b_0|$$
$$+ \log^+ m(2R_k, f^{(-1)} - b_0) + \log^+ m \left(2R_k, \frac{1}{f^{(-1)} - b_0} \right)$$
$$+ \log^+ \frac{1}{R_k} + \log^+ \frac{1}{d} + \log^+ n(2R_k, f^{(-1)} = b_0) + 5 \log 2.$$

Since the order of $f(z)$ equals λ, so does that of $f^{(-1)}(z)$. When R_k is sufficiently large, we have

$$\log^+ m(2R_k, f^{(-1)} - b_0) = O(\log R_k),$$
$$\log^+ m \left(2R_k, \frac{1}{f^{(-1)} - b_0} \right) = O(\log R_k),$$
$$\log^+ \frac{1}{d} = O(\log R_k)$$

and

$$\log^+ n(2R_k, f^{(-1)} = b_0) = O(\log R_k).$$

In view of these estimates and $\mu > 0$, we obtain

$$(5.17) \qquad \log |f(R_k e^{i\theta})| < -(\delta/8)T(R_k, f)$$

for $\theta \in E_k^{(-1)}$

Applying Lemma 3, it follows from (5.16) and (5.17) that

$$\log |f(z)| < -(C_0/2)T(R_k, f)$$

and

$$\log |f^{(j)}(z)| < -\frac{C_0}{2} T(R_k, f) + \log \left\{ j! \left(\frac{20 \cdot 10^{4Q}}{\alpha R_k} \right)^j \right\}$$

in the region D_k defined by (5.4).

When $\theta_0 \in E_k^{(-1)}$ and $z_0 = R_k e^{i\theta_0}$, we have

$$(5.18) \qquad |f^{(-1)}(z) - b_0| \le |f^{(-1)}(z_0) - b_0| + \left| \int_{\overline{z_0 z}} f(\varsigma) \, d\varsigma \right|$$

for $z \in D_k$. Thus (5.13) follows immediately from (5.12), (5.14) and (5.18).

REMARK. Replacing $f^{(-1)}(z)$ by $f^{(l)}(z)$, where l is a negative integer, we can obtain a similar lemma.

6. Estimation of harmonic measure.
6.1. We need some estimates of harmonic measure.

LEMMA 5. *Let α and Q be two numbers with $0 < \alpha < \pi/6$ and $Q > 1$. Putting*

$$D: (10^{-2Q}R_k < |z| < 10^{2Q}R_k) \cap (\theta - \alpha < \arg z < \theta + 5\alpha)$$

and

$$\Gamma_1: (10^{-2Q}R_k < |z| < 10^{2Q}R_k) \cap (\arg z = \theta - \alpha),$$

then $\omega(z_0, \Gamma_1; D) \geq \frac{1}{4}$ for any point z_0 on the arc $(|z| = R_k) \cap (\theta - \alpha < \arg z < \theta + 2\alpha)$.

PROOF. Write

$$\Gamma_2: (10^{-2Q}R_k < |z| < 10^{2Q}R_k) \cap (\arg z = \theta + 5\alpha),$$
$$\Gamma_3: (|z| = 10^{-2Q}R_k) \cap (\theta - \alpha < \arg z < \theta + 5\alpha)$$

and

$$\Gamma_4: (|z| = 10^{2Q}R_k) \cap (\theta - \alpha < \arg z < \theta + 5\alpha).$$

It is clear that

$$(6.1) \qquad \sum_{j=1}^{4} \omega(z_0, \Gamma_j; D) = 1.$$

Further, denoting $D_1: \theta - \alpha < \arg z < \theta + 5\alpha$ and $\Gamma_2': \arg z = \theta + 5\alpha$, we have

$$(6.2) \qquad \begin{aligned} \omega(z_0, \Gamma_2; D) &\leq \omega(z_0, \Gamma_2; D_1) \leq \omega(z_0, \Gamma_2'; D_1) \\ &= \frac{\theta_0 - (\theta - \alpha)}{(\theta + 5\alpha) - (\theta - \alpha)} \leq \frac{1}{2}, \end{aligned}$$

where $\theta_0 = \arg z_0$. From this estimate of harmonic measure, we know that

$$(6.3) \qquad \omega(z_0, \Gamma_3; D) \leq \frac{4((10^{-2Q}R_k)/|z_0|)^{\pi/6\alpha}}{\pi\{1 - (10^{-2Q}R_k)/|z_0|\}^{2\pi/6\alpha}} < \frac{1}{8}$$

and

$$(6.4) \qquad \omega(z_0, \Gamma_4; D) < \tfrac{1}{8}.$$

Substituting (6.2), (6.3) and (6.4) into (6.1), we derive the conclusion of Lemma 5.

6.2. LEMMA 6. *Let α and Q be two numbers with $0 < \alpha < \pi/12$ and $Q > 1$. Denoting*

$$D: (10^{-4Q}R_k < |z| < 10^{4Q}R_k) \cap (\theta - \alpha < \arg z < \theta + 11\alpha)$$

and

$$\Gamma_1: (10^{-4Q}R_k < |z| < 10^{4Q}R_k) \cap (\arg z = \theta - \alpha),$$

we have $\omega(z_0, \Gamma_1; D) \geq \frac{1}{4}$ for any point in the region

$$(10^{-2Q}R_k < |z| < 10^{2Q}R_k) \cap (\theta - \alpha < \arg z < \theta + 5\alpha).$$

PROOF. In a similar way to the proof of Lemma 5, we write

$$\Gamma_2 : (10^{-4Q}R_k < |z| < 10^{4Q}R_k) \cap (\arg z = \theta + 11\alpha),$$
$$\Gamma_3 : (|z| = 10^{-4Q}R_k) \cap (\theta - \alpha < \arg z < \theta + 11\alpha),$$
$$\Gamma_4 : (|z| = 10^{4Q}R_k) \cap (\theta - \alpha < \arg z < \theta + 11\alpha),$$
$$D_1 : \theta - \alpha < \arg z < \theta + 11\alpha$$

and $\Gamma_2' : \arg z = \theta + 11\alpha$. Thus we have

$$\omega(z_0, \Gamma_2; D) \le \omega(z_0, \Gamma_2'; D_1) \le \tfrac{1}{2}, \quad \omega(z_0, \Gamma_3; D) < \tfrac{1}{8}$$

and $\omega(z_0, \Gamma_4; D) < \tfrac{1}{8}$. Lemma 6 follows from these estimates and the fact that $\sum_{j=1}^{4} \omega(z_0, \Gamma_j; D) = 1$.

6.3. Finally we give an estimation between the maximum modulus and characteristic function.

LEMMA 7. *Suppose that $f(z)$ satisfies the assumptions of Lemma 2 with $p_0 + p_{-1} \ge 1$. Then a positive number V can be chosen sufficiently large such that*

(6.5)
$$\max\{\log M(10^{-4V}R_k, f), \log M(10^{-4V}R_k, f^{(-1)})\}$$
$$< \tfrac{1}{4}(C_0/4)^{q+2} T(R_k, f),$$

where $\{R_k\}$ is defined in Lemma 2, q denotes the number of Borel directions of $f(z)$ and C_0 is given by (5.5).

PROOF. When $f(z)$ has a finite deficient value a_1, according to a result of A. Weitsman [11], there is a positive number $\tau = \tau(\delta(a_1, f), \delta(\infty, f))$ such that the inequality

$$\varliminf_{t \to +\infty} \frac{T(\sigma t, f)}{T(t, f)} \ge \sigma^{\tau}$$

holds for any $\sigma \ge 1$. When $p_0 = 0$ and $f^{(-1)}(z)$ has a finite deficient value b_1, then $\delta(b_1, f^{(-1)}) \le \delta(0, f)$ and there is $\tau_1 = \tau_1(\delta(0, f), \delta(\infty, f)) > 0$ with

$$\varliminf_{t \to +\infty} \frac{T(\sigma t, f)}{T(t, f)} \ge \sigma^{\tau_1}.$$

Setting $\tau^* = \min(\tau, \tau_1)$, we have

$$\log M(10^{-4V}R_k, f) \le 3T(2 \cdot 10^{-4V}R_k, f) < 4(2 \cdot 10^{-4V})^{\tau^*} T(R_k, f).$$

Since

$$\log M(r, f^{(-1)}) \le \log |f^{(-1)}(0)| + \log 2r + \log M(r, f)$$
$$\le O(\log r) + 3T(2r, f),$$

it is easy to see that

$$\log M(10^{-4V}R_k, f^{(-1)}) \le O(\log R_k) + 3T(2 \cdot 10^{-4V}R_k, f)$$
$$< 4(2 \cdot 10^{-4V})^{\tau^*} T(R_k, f).$$

Therefore, if we choose

(6.6)
$$V > \frac{1}{4 \log 10} \left\{ \left(1 + \frac{4}{\tau^*}\right) \log 2 + \frac{q+2}{\tau^*} \log \frac{4}{C_0} \right\},$$

then (6.5) follows immediately.

REMARK. In general, if L is a negative integer, we can establish a similar lemma to prove that

$$\max_{L \le j \le 0} \{\log M(10^{-4V} R_k, f^{(j)})\} < \frac{1}{4} \left(\frac{C_0}{4}\right)^{q+2} T(R_k, f).$$

7. Proof of Theorem.

7.1. Now we prove the theorem formulated in the introduction.

When $\mu \le \frac{1}{2}$, the lower orders of $f^{(l)}(z)$ $(l = 0, -1, -2, \ldots)$ are all equal to μ. Since an entire function of lower order less than or equal to $\frac{1}{2}$ cannot possess any finite deficient value, the conclusion of the theorem is clear.

When $\mu > \frac{1}{2}$, we note first by Lemma B that the order λ of $f(z)$ must be finite, because the number of its Borel directions of order $\ge \mu$ is finite. For simplicity of notation, we only prove $p_0 + p_{-1} \le 2\mu$. The same procedure can derive the general conclusion $\sum_{l=0}^{-\infty} p_l \le 2\mu$.

Suppose that $1 \le p_0 < +\infty$ and $1 \le p_{-1} < +\infty$. Let a_j $(j = 1, 2, \ldots, p_0)$ be the deficient values of $f(z)$ with deficiencies $\delta(a_j, f)$, b_l $(l = 1, 2, \ldots, p_{-1})$ be the deficient values of $f^{(-1)}(z)$ with deficiencies $\delta(b_l, f^{(-1)})$ and

$$\delta = \min_{\substack{1 \le j \le p_0 \\ 1 \le l \le p_{-1}}} \{\delta(a_j, f), \delta(b_l, f^{(-1)})\}.$$

From Lemma 2, there are two sequences $\{t_k\}$ and $\{R_k\}$ such that

$$t_k \le R_k \le 2t_k, \qquad \lim_{k \to +\infty} \frac{\log T(t_k, f)}{\log t_k} = \mu,$$

$$\text{mes } E\{\theta \colon \log |f(R_k e^{i\theta}) - a_j| \le -(\delta/4) T(R_k, f)\} > B, \qquad j = 1, 2, \ldots, p_0,$$

and

$$\text{mes } E\{\theta \colon \log |f^{(-1)}(R_k e^{i\theta}) - b_l| \le -(\delta/4) T(R_k, f)\} > B, \qquad l = 1, 2, \ldots, p_{-1},$$

where $B = B(\delta, \mu, p_0, p_{-1}) > 0$ is determined by (4.4).

Denote by $\arg z = \theta_\nu$ $(\nu = 1, 2, \ldots, q, 0 \le \theta_1 < \theta_2 < \cdots < \theta_q < 2\pi)$ the q Borel directions of order $\ge \mu$ of $f(z)$ which divide the plane into q sectors. Let

$$\omega = \min_{1 \le \nu \le q} (\theta_{\nu+1} - \theta_\nu), \qquad \theta_{q+1} = 2\pi + \theta_1.$$

Choose a fixed integer k, sufficiently large, so that corresponding to a_1 there is an angle, say $\theta_{\nu_1} < \arg z < \theta_{\nu_1+1}$, such that

$$\text{mes } E\{\theta \colon \theta_{\nu_1} < \theta < \theta_{\nu_1+1}, \log |f(R_k e^{i\theta}) - a_1| \le -(\delta/4) T(R_k, f)\} > B/q.$$

Hereinafter we denote by $A(r_1, r_2; \varphi_1, \varphi_2)$ the region $(r_1 < |z| < r_2) \cap (\varphi_1 < \arg z < \varphi_2)$ and $\overline{A}(r_1, r_2; \varphi_1, \varphi_2)$ its closure.

With two positive numbers $\alpha < \min(K/4q, \omega/24)$ and $Q > 1$ chosen, applying Lemma 3 to $f(z)$ and a_1 yields that

(7.1) $\log |f(z) - a_1| < -C_0 T(R_k, f)$

in the region $\overline{A}(10^{-4Q} R_k, 10^{4Q} R_k; \theta_{\nu_1} + \alpha, \theta_{\nu_1+1} - \alpha)$, where C_0 is given by (5.5).

Similarly, corresponding to a_2 there is an angle, say $\theta_{\nu_2} < \arg z < \theta_{\nu_2+1}$, such that

$$\text{mes } E\{\theta \colon \theta_{\nu_2} < \theta < \theta_{\nu_2+1}, \log |f(R_k e^{i\theta}) - a_2| \le -(\delta/4) T(R_k, f)\} > B/q.$$

Hence the inequality

(7.2) $$\log |f(z) - a_2| < -C_0 T(R_k, f)$$

holds in the region $\overline{A}(10^{-4Q} R_k, 10^{4Q} R_k; \theta_{\nu_2} + \alpha, \theta_{\nu_2+1} - \alpha)$. It is easy to see that the angles $\theta_{\nu_1} < \arg z < \theta_{\nu_1+1}$ and $\theta_{\nu_2} < \arg z < \theta_{\nu_2+1}$ have no common point.

By this procedure, there are $p_0 + p_{-1}$ angles, one corresponding to each of a_j and b_l, $\theta_{\nu_n} < \arg z < \theta_{\nu_n+1}$,

$$0 \le \theta_{\nu_1} < \theta_{\nu_1+1} \le \theta_{\nu_2} < \theta_{\nu_2+1} \le \cdots < \theta_{\nu_{p_0+p_{-1}}+1} \le \theta_1 + 2\pi,$$

and in every angle $\theta_{\nu_n} < \arg z < \theta_{\nu_n+1}$, one an only one among the inequalities

(7.3) $$\log |f(z) - a_j| < -C_0 T(R_k, f), \qquad j = 1, 2, \ldots, p_0,$$

and

(7.4) $$\log |f^{(-1)}(z) - b_l| < -C_0 T(R_k, f), \qquad l = 1, 2, \ldots, p_{-1},$$

holds uniformly in the region $\overline{A}(10^{-4Q} R_k, 10^{4Q} R_k; \theta_{\nu_n} + \alpha, \theta_{\nu_n+1} - \alpha)$.

7.2. For two angles $\theta_{\nu_1} < \arg z < \theta_{\nu_1+1}$ and $\theta_{\nu_2} < \arg z < \theta_{\nu_2+1}$ corresponding to a_1 and a_2 respectively, we conclude that there is a point z_1, satisfying the condition $\log |f(z_1) - a_1| > 0$ in the region $\overline{A}(10^{-2Q} R_k, 10^{2Q} R_k; \theta_{\nu_1+1} - \alpha, \theta_{\nu_2} + 5\alpha)$.

In fact, if this assertion were not true, then the inequality

(7.5) $$\log |f(z) - a_1| \le 0$$

would hold uniformly in $\overline{A}(10^{-2Q} R_k, 10^{2Q} R_k; \theta_{\nu_1+1} - \alpha, \theta_{\nu_2} + 5\alpha)$. Let

$$D_{k1}: A(10^{-2Q} R_k, 10^{2Q} R_k; \theta_{\nu_1+1} - \alpha, \theta_{\nu_1+1} + 5\alpha),$$

$$\Gamma_1: (10^{-2Q} R_k < |z| < 10^{2Q} R_k) \cap (\arg z = \theta_{\nu_1+1} - \alpha)$$

and $C\Gamma_1$ be the rest of the boundary of D_{k1}. For an arbitrary point z_0 on the arc $(|z| = R_k) \cap (\theta_{\nu_1+1} - \alpha < \arg z < \theta_{\nu_1+1} + 2\alpha)$, we obtain

$$\log |f(z_0) - a_1| \le \omega(z_0, \Gamma_1; D_{k1}) \max_{z \in \Gamma_1} \log |f(z) - a_1|$$

$$+ \{1 - \omega(z_0, \Gamma_1; D_{k1})\} \max_{z \in C\Gamma_1} \log |f(z) - a_1|$$

$$< -(C_0/4) T(R_k, f),$$

by noting (7.1) on Γ_1, (7.5) and $\omega(z_0, \Gamma_1; D_{k1}) \ge \frac{1}{4}$ from Lemma 5. Thus

$$\text{mes } E\{\theta: \theta_{\nu_1+1} < \theta < \theta_{\nu_1+2}, \log |f(R_k e^{i\theta}) - a_0| < -(C_0/4) T(R_k, f)\} > 2\alpha.$$

Applying Lemma 3 to $f(z)$, a_1 and the angle $\theta_{\nu_1+1} < \arg z < \theta_{\nu_1+2}$ which is the neighbor angle of $\theta_{\nu_1} < \arg z < \theta_{\nu_1+1}$, we get

$$\log |f(z) - a_1| < -(C_0^2/4) T(R_k, f)$$

in $\overline{A}(10^{-4Q} R_k, 10^{4Q} R_k; \theta_{\nu_1+1} + \alpha, \theta_{\nu_1+2} - \alpha)$.

By the same deduction, we have

$$\log |f(z) - a_1| < -(C_0/4)^2 T(R_k, f)$$

on $(|z| = R_k) \cap (\theta_{\nu_1+2} - \alpha < \arg z < \theta_{\nu_1+2} + 2\alpha)$ and

$$\log |f(z) - a_1| < -(C_0/4)^2 C_0 T(R_k, f)$$

in $\overline{A}(10^{-4Q}R_k, 10^{4Q}R_k; \theta_{\nu_1+2} + \alpha, \theta_{\nu_1+3} - \alpha)$.

Applying this procedure in succession through $\nu_2 - \nu_1$ ($\leq q$) times, we obtain finally

$$
(7.6) \qquad \begin{aligned}
\log|f(z) - a_1| &< -(C_0/4)^{\nu_2-\nu_1} C_0 T(R_k, f) \\
&\leq -(C_0/4)^q C_0 T(R_k, f)
\end{aligned}
$$

in $\overline{A}(10^{-4Q}R_k, 10^{4Q}R_k; \theta_{\nu_2} + \alpha, \theta_{\nu_2+1} - \alpha)$. It is clear that the inequality (7.6) contradicts (7.2).

From the above procedure, we know that a point z_1 in the region

$$\overline{A}(10^{-2Q}R_k, 10^{2Q}R_k; \theta_{\nu_1+h_1} - \alpha, \theta_{\nu_1+h_1} + 5\alpha) \qquad (1 \leq h_1 \leq \nu_2 - \nu_1)$$

can be found such that

$$(7.7) \qquad \log|f(z_1) - a_1| > 0$$

and that in the region $\overline{A}(10^{-2Q}R_k, 10^{2Q}R_k; \theta_{\nu_1+1} - \alpha, \theta_{\nu_1+h_1} - \alpha)$ $\log|f(z) - a_1| \leq 0$ holds uniformly. Thus we have

$$(7.8) \qquad \log|f(z) - a_1| < -C_0(C_0/4)^{h_1} T(R_k, f)$$

on the segment $\Gamma_1': (10^{-4Q}R_k \leq |z| \leq 10^{4Q}R_k) \cap (\arg z = \theta_{\nu_1+h_1} - \alpha)$.

7.3. Further, there exists a point z_1^* on the closure of

$$D_{k1}': A(10^{-4Q}R_k, 10^{4Q}R_k; \theta_{\nu_1+h_1} - \alpha, \theta_{\nu_1+h_1} + 11\alpha)$$

such that

$$(7.9) \qquad \log|f(z_1^*)| > (C_0/4)^{q+2} T(R_k, f),$$

where q is the number of Borel directions of order $\geq \mu$.

In fact, let

$$(7.10) \qquad |f(z_1^*) - a_1| = \max_{z \in \overline{D}_{k1}'} |f(z) - a_1|,$$

and denote by $C\Gamma_1'$ the complement of Γ_1' with respect to the boundary of D_{k1}'. We obtain from (7.7), (7.8), (7.10) and Lemma 6 that

$$
\begin{aligned}
0 < \log|f(z_1) - a_1| &< \omega(z_1, \Gamma_1', D_{k1}') \log M(\Gamma_1', f - a_1) \\
&\quad + \{1 - \omega(z_1, \Gamma_1', D_{k1}')\} \log M(C\Gamma_1', f - a_1) \\
&< \tfrac{1}{4}\{-C_0(C_0/4)^{h_1} T(R_k, f)\} + \log|f(z_1^*) - a_1|.
\end{aligned}
$$

Therefore the inequality (7.9) follows immediately.

If the angle $\theta_{\nu_3} < \arg z < \theta_{\nu_3+1}$ corresponds to $f^{(-1)}(z)$ and the deficient value b_1, then we have by Lemma 4 that $\log|f^{(-1)}(z) - b_1| < -C_0 T(R_k, f)$ and $\log|f(z)| < -(C_0/2)T(R_k, f)$ in the region $\overline{A}(10^{-4Q}R_k, 10^{4Q}R_k; \theta_{\nu_3}+\alpha, \theta_{\nu_3+1}-\alpha)$. By the same deduction, we obtain a point z_2^* in

$$\overline{A}(10^{-4Q}R_k, 10^{4Q}R_k; \theta_{\nu_2+h_2} - \alpha, \theta_{\nu_2+h_2} + 11\alpha) \qquad (1 \leq h_2 \leq \nu_3 - \nu_2)$$

with $\log|f(z_2^*)| > (C_0/4)^{q+2} T(R_k, f)$.

If the angle $\theta_{\nu_4} < \arg z < \theta_{\nu_4+1}$ corresponds to $f^{(-1)}(z)$ and the deficient value b_2, then we also obtain a point z_3^* in

$$\overline{A}(10^{-4Q}R_k, 10^{4Q}R_k; \theta_{\nu_3+h_3} - \alpha, \theta_{\nu_3+h_3} + 11\alpha) \qquad (1 \leq h_3 \leq \nu_4 - \nu_3)$$

with $\log|f^{(-1)}(z_3^*)| > (C_0/4)^{q+2} T(R_k, f)$.

In general, there is a point z_n^* in

$$\overline{A}(10^{-4Q}R_k, 10^{4Q}R_k, \theta_{\nu_n+h_n} - \alpha, \theta_{\nu_n+h_n} + 11\alpha) \qquad (1 \le h_n \le \nu_{n+1} - \nu_n)$$

satisfying either

(7.11) $$\log |f(z_n^*)| > (C_0/4)^{q+2} T(R_k, f)$$

or

(7.12) $$\log |f^{(-1)}(z_n^*)| > (C_0/4)^{q+2} T(R_k, f).$$

7.4. If C_0 is given by (5.5), we can find a number C_0' such that

(7.13) $$\tfrac{1}{4}(C_0/4)^{q+2} \le C_0' \le \tfrac{1}{2}(C_0/4)^{q+2}$$

and that $f'(z)$ has no zero on the level curves $|f(z)| = C_0'T(R_k, f)$ which are thus analytic.

Consider the set

(7.14) $$\Omega: \{z: \ \log |f(z)| > C_0'T(R_k, f), |z| < 10^{4V}R_k\},$$

where V is given by (6.6). Let Ω_{k1} be a component of Ω which contains the point z_1^*. It is clear from Lemma 7, (7.1) and (7.3) that Ω_{k1} does not meet $|z| = 10^{-4V}R_k$, $\arg z = \theta_{\nu_1+1} - \alpha$ and $\arg z = \theta_{\nu_2} + \alpha$. But $\overline{\Omega}_{k1} \cap (|z| = 10^{4V}R_k)$ contains at least one arc by the maximum modulus theorem.

Let $r\varphi_{k1}(r)$ be the linear measure of $\Omega_{k1} \cap (|z| = r)$ and let Γ_{k1} be $\overline{\Omega}_{k1} \cap (|z| = 10^{4V}R_k)$ and $C\Gamma_{k1}$ be the complement of Γ_{k1} with respect to the boundary of Ω_{k1}. When $|z_1^*| \le r < 10^{4V}R_k$, we have

$$\log |f(z_1^*)| \le \{1 - \omega(z_1^*, \Gamma_{k1}; \Omega_{k1})\} \max_{z \in C\Gamma_{k1}} \log |f(z)|$$
$$+ \omega(z_1^*, \Gamma_{k1}; \Omega_{k1}) \max_{z \in \Gamma_{k1}} \log |f(z)|.$$

In view of (7.9), (7.13), (7.14) and an estimation [8, p. 116] of harmonic measure, we obtain

$$(C_0/4)^{q+2}T(R_k, f) < \log |f(z_1^*)| < \tfrac{1}{2}(C_0/4)^{q+2}T(R_k, f)$$
$$+ \log M(10^{4V}R_k, f)9\sqrt{2} \exp\left\{ -\pi \int_{2|z_1^*|}^{10^{4V}R_k/2} \frac{dr}{r\varphi_{k1}(r)} \right\}.$$

Thus

(7.15) $$\pi \int_{2\cdot10^{4Q}R_k}^{10^{4V}R_k/2} \frac{dr}{r\varphi_{k1}(r)} < \log T(2 \cdot 10^{4V}R_k, f)$$
$$- \log T(R_k, f) + \log\{54\sqrt{2}(4/C_0)^{q+2}\}.$$

In the general case, if (7.11) holds, then there is a component Ω_{kn} of Ω which contains the point z_n^* and does not meet $|z| = 10^{-4V}R_k$, $\arg z = \theta_{\nu_n+1} - \alpha$ and $\arg z = \theta_{\nu_n+1} + \alpha$. Let $r\varphi_{kn}(r)$ be the linear measure of $\Omega_{kn} \cap (|z| = r)$. Then we have the same inequality (7.15) with $\varphi_{kn}(r)$ $(n = 1, 2, \ldots, n_1)$.

Similarly, there is $C_0''T(R_k, f)$ in

$$[\tfrac{1}{4}(C_0/4)^{q+2}T(R_k, f), \tfrac{1}{2}(C_0/4)^{q+2}T(R_k, f)]$$

such that $f(z)$ has no zero on the level curves $|f^{(-1)}(z)| = C_0'' T(R_k, f)$ which are thus analytic.

Consider the set

$$\Omega': \{z: \log|f^{(-1)}(z)| > C_0'' T(R_k, f), |z| < 10^{4V} R_k\}.$$

If (7.12) holds, let Ω'_{kn} be a component of Ω' which contains the point z_n^*. We can see that Ω'_{kn} does not meet $|z| = 10^{-4V} R_k$, $\arg z = \theta_{\nu_n+1} - \alpha$ and $\arg z = \theta_{\nu_n+1} + \alpha$. Let $r\varphi_{kn}^*(r)$ be the linear measure of $(|z| = r) \cap \Omega'_{kn}$. Then, if $|z_n^*| \le r < 10^{4V} R_k$, we have

$$(C_0/4)^{q+2} T(R_k, f^{(-1)}) < \log|f^{(-1)}(z_n^*)| < \tfrac{1}{2}(C_0/4)^{q+2} T(R_k, f^{(-1)})$$
$$+ \log M(10^{4V} R_k, f^{(-1)}) \cdot 9\sqrt{2} \exp\left\{-\pi \int_{2|z_n^*|}^{10^{4V} R_k/2} \frac{dr}{r\varphi_{kn}^*(r)}\right\}.$$

Since

$$\log M(10^{4V} R_k, f^{(-1)}) \le \log M(10^{4V} R_k, f) + O(\log 10^{4V} R_k)$$
$$< 4T(2 \cdot 10^{4V} R_k, f)$$

and

$$T(R_k, f) \le T(R_k, f^{(-1)}) + m\left(R_k, \frac{f}{f^{(-1)}}\right)$$
$$= T(R_k, f^{(-1)}) + O(\log R_k) < \tfrac{4}{3} T(R_k, f^{(-1)}),$$

this yields that

$$\pi \int_{2 \cdot 10^{4Q} R_k}^{10^{4V} R_k/2} \frac{dr}{r\varphi_{kn}^*(r)} < \log T(2 \cdot 10^{4V} R_k, f)$$
(7.16)
$$- \log T(R_k, f) + \log\left\{144\sqrt{2}\left(\frac{4}{C_0}\right)^{q+2}\right\}.$$
$$n = 1, 2, \dots, n_2.$$

Therefore we obtain finally n_1 components Ω_{kn} $(n = 1, 2, \dots, n_1)$ of Ω and n_2 components Ω'_{kn} $(n = 1, 2, \dots, n_2)$ of Ω' and the corresponding inequalities (7.15) with $\varphi_{kn}(r)$ and (7.16). It is very important to note that $n_1 + n_2 = p_0 + p_{-1}$ and that any one of Ω_{kn} $(n = 1, 2, \dots, n_1)$ and Ω'_{kn} $(n = 1, 2, \dots, n_2)$ is distant from all others. Thus

$$\sum_{n=1}^{n_1} \varphi_{kn}(r) + \sum_{n=1}^{n_2} \varphi_{kn}^*(r) \le 2\pi,$$

so that

$$(p_0 + p_{-1})^2 = \left\{\sum_{n=1}^{n_1} (\varphi_{kn}(r))^{1/2} \frac{1}{(\varphi_{kn}(r))^{1/2}} + \sum_{n=1}^{n_2} (\varphi_{kn}^*(r))^{1/2} \frac{1}{(\varphi_{kn}^*(r))^{1/2}}\right\}^2$$
$$\le \left\{\sum_{n=1}^{n_1} \varphi_{kn}(r) + \sum_{n=1}^{n_2} \varphi_{kn}^*(r)\right\}\left\{\sum_{n=1}^{n_1} \frac{1}{\varphi_{kn}(r)} + \sum_{n=1}^{n_2} \frac{1}{\varphi_{kn}^*(r)}\right\}$$
$$\le 2\pi\left\{\sum_{n=1}^{n_1} \frac{1}{\varphi_{kn}(r)} + \sum_{n=1}^{n_2} \frac{1}{\varphi_{kn}^*(r)}\right\}.$$

Consequently, we have

$$
\frac{(p_0 + p_{-1})^2}{2} \int_{2 \cdot 10^{4Q} R_k}^{10^{4V} R_k/2} \frac{dr}{r}
$$

$$
\leq \sum_{n=1}^{n_1} \pi \int_{2 \cdot 10^{4Q} R_k}^{10^{4V} R_k/2} \frac{dr}{r \varphi_{kn}(r)} + \sum_{n=1}^{n_2} \pi \int_{2 \cdot 10^{4Q} R_k}^{10^{4V} R_k/2} \frac{dr}{r \varphi_{kn}^*(r)}
$$

$$
\leq (p_0 + p_{-1})\{\log T(2 \cdot 10^{4V} R_k, f) - \log T(R_k, f)\}
$$

$$
+ (p_0 + p_{-1}) \log\{144\sqrt{2}(4/C_0)^{q+2}\}.
$$

But

$$
t_k \leq R_k \leq 2t_k, \qquad T(4 \cdot 10^{4V} t_k, f) < \{(8 \cdot 4 \cdot 10^{4V})^\mu + o(1)\} T(t_k, f)
$$

and $T(R_k, f) \geq T(t_k, f)$ by Lemma 2 and Lemma 1 with $\beta_1 = 8$, $\beta_2 = 4 \cdot 10^{4V}$ and therefore we obtain

$$
(p_0 + p_{-1}) \log \frac{10^{4V}}{4 \cdot 10^{4Q}} \leq 2 \log(33 \cdot 10^{4V})^\mu + 2 \log \left\{ 144\sqrt{2} \left(\frac{4}{C_0} \right)^{q+2} \right\}.
$$

Letting $V \to +\infty$, we have $p_0 + p_{-1} \leq 2\mu$.

If p_0 or p_{-1} equals $+\infty$, then we can choose p_0' deficient values of $f(z)$ and p_{-1}' deficient values of $f^{(-1)}(z)$ such that $p_0' + p_{-1}' = [2\mu] + 1$. The same procedure, however, leads to $p_0' + p_{-1}' \leq 2\mu$. This contradiction shows that both p_0 and p_{-1} must be finite. When $p_0 = 0$ or $p_{-1} = 0$, the above deduction is also effective. Thus the proof of the theorem is complete.

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