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J. Differential Equations 263 (2017) 7000–7097

Journal of Differential Equations

www.elsevier.com/locate/jde

The Fučík spectrum of Schrödinger operator and the existence of four solutions of Schrödinger equations with jumping nonlinearities

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Received 15 June 2017; revised 22 July 2017
Available online 9 August 2017

Abstract

This paper contains the existence of four solutions of Schrödinger equations with jumping nonlinearities. The proof procedure is supported by a lot of new results. Initially, a consequence is rendered as a minimax principle on $H^1\left(\mathbb{R}^N\right)$, which allows us to achieve the feasibility verification of the (PS) condition. Furthermore, the constructions of minimal and maximal curves of Fučík spectrum in Q_l (see the introduction for the definition of Q_l) warrant an intensive investigation. That we encounter some thorny problems is largely due to the absence of compact embedding and the appearance of essential spectrum. Based on a nontrivial argument, we can compute critical groups of homogeneous functional at zero if (a, b) is free of Fučík spectrum and $(a, b) \in Q_l$. This together with convexity and concavity offers a detailed description of the two curves by a series of sophisticated tricks. Additionally, we present a new version of Morse theory in view of the fact that classical version doesn't work directly for weak smooth functional on $H^1\left(\mathbb{R}^N\right)$. Finally, we prove a weak maximum principle for $\mathbb{R}^N$, which serves as a tool to get a critical point in positive and negative cone respectively and also compute critical groups of critical points of mountain pass type. With the help of above preparations, we attain the ultimate aim by Morse inequalities and various exact homology sequences.

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1 The first author is supported by NSFC (11471319) and BCMIIS.
2 The second author is supported by NSFC (11271353) and NSFC (11471319).

http://dx.doi.org/10.1016/j.jde.2017.07.038
0022-0396/

MSC: 35P05; 35A15

Keywords: Schrödinger equation; Minimax principle; Morse theory; Maximum principle; Exact homology sequences

Contents

1. Introduction

This paper is mainly concerned with nonlinear Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x,u), & x\in\mathbb{R}^N, \\ u(x)\to 0, & \text{as } |x|\to+\infty, \end{cases} \tag{1.1}$$

arising from study of standing wave solutions of time-dependent nonlinear Schrödinger equations. The corresponding energetic functional of (1.1) is of the form as:

$$J(u) = J(u,a,b) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 + V(x)u^2\right)dx - \int_{\mathbb{R}^N} F(x,u),$$

where $F(x,u)=\int_0^u f(x,s)\,ds$, $f(x,u)$ is a Carathéodory function on $\mathbb{R}^N\times\mathbb{R}$ such that

$$\frac{f(x,t)}{t}\to\begin{cases} a, \text{ as } t\to-\infty, \\ b, \text{ as } t\to+\infty, \end{cases} \tag{1.2}$$

$(a, b) \notin \Sigma(-\Delta + V)$, and $\Sigma(-\Delta + V)$ is called Fučík spectrum, defined as the set of all $(a, b) \in \mathbb{R}^2$ such that

$$\begin{cases} -\Delta u + V(x)u = au^- + bu^+, & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty, \end{cases} \tag{1.3}$$

has a nontrivial solution u in the form domain of $-\Delta + V$, where $u^+ = \max\{u, 0\}$, $u^- = \min\{u, 0\}$, and the form domain of $-\Delta + V$ is the Hilbert space $H^1(\mathbb{R}^N)$ equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2 \right)^{\frac{1}{2}}$$

and the inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + uv.$$

From the variational point of view, the solutions of (1.3) are the critical points of the following energetic functional

$$I(u) = I(u, a, b) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 - a\left|u^-\right|^2 - b\left|u^+\right|^2, u \in H^1\left(\mathbb{R}^N\right).$$

Throughout the paper, we always assume that the linear potential V is real-valued, and $V = V_1 + V_2$, $V_1 \in L^p(\mathbb{R}^N)$, $V_2 \in L^\infty(\mathbb{R}^N)$, with $p = 2$ if $N \le 3$, $p > 2$ if $N = 4$ and $p > \frac{N}{2}$ if $N \ge 5$.

The Fučík spectrum was originally introduced in the 1970s by Fučík [11] and Dancer [10], and defined on a smooth bounded domain $\Omega \subset \mathbb{R}^N$. Denote by $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ the distinct eigenvalues of $-\Delta$ with Dirichlet boundary condition. It is then clear that $(\lambda_l, \lambda_l) \in \Sigma(-\Delta)$ for any $l \in \mathbb{N}$ and $(\{\lambda_1\} \times \mathbb{R}) \cup (\mathbb{R} \times \{\lambda_1\}) \subset \Sigma(-\Delta)$. Put $Q_l = (\lambda_{l-1}, \lambda_{l+1})^2$ for $l \ge 2$. Schechter in [35], [38] constructed two decreasing curves C_{l1}, C_{l2} (which may coincide) in Q_l passing through (λ_l, λ_l) such that all points on the curves are in $\Sigma(-\Delta)$, while points in Q_l that are above both curves or below both curves are not in $\Sigma(-\Delta)$. When the curves do not coincide, the region between them is called a type (II) region, which may or may not intersect $\Sigma(-\Delta)$ (see also [34] and [37]).

Concerning the Fučík spectrum for Schrödinger operator on $\mathbb{R}^N$, to the best of our knowledge, so far the unique work appeared on [5], in which the authors gave a full characterization of the first nontrivial curve for certain types of potential by minimax methods.

Deeply attracted by the constructions of Fučík spectrum curves emanating from (λ, λ), $\lambda \in \sigma_{\text{dis}}(-\Delta + V)$, in this paper we are devoted to making a complete description of C_{l1}, C_{l2} in Q_l for the case $\lambda_{l-1}, \lambda_l, \lambda_{l+1} \in \sigma_{\text{dis}}(-\Delta + V)$, $\lambda_{l-1} < \lambda_l < \lambda_{l+1} < \inf \sigma_{\text{ess}}(-\Delta + V)$, $l \ge 3$. However, it seems that we can not expect to achieve the target merely by a simple extension of [35] and [38] since the main difficulty, to a great extent, has arisen from the absence of compact embedding and the appearance of essential spectrum. For this reason, we introduce a largely different measure, which does the trick (see Sections 4 and 5 for more details).

Another challenge comes from validation of compactness condition. In [25], the authors dealt with both nonresonant and resonant case for J (see also [25] for the argument on more general functional). However, for $(a,b) \notin \Sigma(-\Delta+V)$ and $a \neq b$, they verified that J satisfies the (PS) condition with a strong hypothesis $[\Gamma(a,b), \Lambda(a,b)] \cap \sigma(-\Delta+V) = \varnothing$, $\Gamma(a,b) := \min\{a,b\}$, $\Lambda(a,b) := \max\{a,b\}$. In comparison with [25], for $(a,b) \notin \Sigma(-\Delta+V)$ and $a \neq b$, under some certain hypotheses we derive identical result for J when admitting $(\Gamma(a,b), \Lambda(a,b)) \cap \sigma_{\text{dis}}(-\Delta+V) \neq \varnothing$, and a minimax principle on $H^1(\mathbb{R}^N)$ (see Section 2) plays a crucial role in the process of proof. Due to the length limitation of this paper, we do not intend to make a discussion on the resonant case.

Let V be a Kato–Rellich potential (see Definition 2.1) and suppose $\lambda_1 = \inf\sigma(-\Delta+V) < \lambda_2 < \cdots < \lambda_l < \lambda_{l+1} < \sigma_0 := \inf\sigma_{\text{ess}}(-\Delta+V)$, $\sigma_{\text{dis}}(-\Delta+V) \cap [\lambda_1, \lambda_{l+1}] = \{\lambda_i\}_{i=1}^{l+1}$, $l \geq 2$. Denote

$$Q^*_{l+1} := \left\{(a,b) \in \mathbb{R}^2 : \lambda_l \leq \Gamma(a,b) \leq \Lambda(a,b) \leq \widetilde{\theta}_{l+1} = \Gamma\left(\mu_{d_{l+1}+1}, \sigma_0\right)\right\};$$

$$C^{(1)}_{k+1} := \left\{(a,b) \in Q^*_{k+1} : b = \widetilde{\nu}_k(a)\right\};$$

$$C^{(2)}_{k+1} := \left\{(a,b) \in Q^*_{k+1} : b = \widetilde{\mu}_{k+1}(a)\right\},$$

where $\mu_j = \mu_j(-\Delta+V)$ (see Proposition 2.8), d_{l+1} is the sum of multiplicity of $\lambda_1, \cdots, \lambda_{l+1}$, and make the following hypotheses:

(V_2) $V(x)$ is a real potential, s.t.,

$$\inf_{x\in\mathbb{R}^N} V(x) > -\infty.$$

(V_3) $V \in C^{N-2,\alpha}\left(\mathbb{R}^N\right)$ if $N \geq 3$ and $V \in C^1\left(\mathbb{R}^N\right)$ if $N = 1, 2$, $0 < \alpha < 1$;

(V_4) $\lambda_1 = \inf\sigma(-\Delta+V) < \lambda_2 < \cdots < \lambda_l < \sigma_0$, $\sigma_{\text{dis}}(-\Delta+V) \cap [\lambda_1, \sigma_0) = \{\lambda_i\}_{i=1}^{l}$;

(f_1) Set $f(x,s) = as^- + bs^+ + g(x,s)$, $g(x,0) = 0$, $\lim\limits_{|s|\to\infty} \frac{g(x,s)}{s} = 0$ uniformly with respect to $x \in \mathbb{R}^N$, s.t., $\exists\beta > 0$, such that $\forall s_1, s_2 \in \mathbb{R}$, $\forall x \in \mathbb{R}^N$,

$$|g(x,s_1) - g(x,s_2)| \leq \beta|s_1 - s_2|$$

and $\beta < \sigma_0 - \Lambda(a,b)$, where $\Lambda(a,b) = \max\{a,b\}$.

(f_3) Let $f(x,s) = a_0 s^- + b_0 s^+ + \widetilde{g}(x,s)$, $\lim\limits_{s\to 0_-} \frac{f(x,s)}{s} = a_0$, $\lim\limits_{s\to 0_+} \frac{f(x,s)}{s} = b_0$, uniformly on $x \in \mathbb{R}^N$, and $\widetilde{g}(x,s) \in C^{N-2,\alpha}\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$ if $N \geq 3$, and $\widetilde{g}(x,s) \in C^1\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$ if $N = 1, 2$;

(f_4) $\lambda_k < \Gamma(a,b) \leq \Lambda(a,b) < \widetilde{\theta}_{k+1} = \Gamma\left(\mu_{d_{k+1}+1}, \sigma_0\right)$, $l > k \geq 3$;

(f_5) $\Lambda(a_0, b_0) < \lambda_1$;

(f_6) $f'_s(x,s) > \frac{f(x,s)}{s} > -m$, $\forall s \neq 0$, a.e. on $x \in \mathbb{R}^N$, where m is given by Section 3.

The hypothesis (f_1) is due to Section 3 (see p. 7014). The hypothesis (V_2) is offered by Section 5.1 (see p. 7027). The definitions of Q^*_{k+1}, $\widetilde{\nu}_k(a)$, $\widetilde{\mu}_{k+1}(a)$ are presented by Section 5.2

(see p. 7042, p. 7048 and p. 7058 respectively). As to the hypotheses (V_3), (V_4), (f_3), (f_4), (f_5), (f_6), please refer to Section 7 (see p. 7069).

With the aid of the above mentioned hypotheses, our main result concerning the existence of four nontrivial solutions of (1.1) reads:

Theorem 1.1. *Let V be a Kato–Rellich potential. Suppose that (V_2)–(V_4), (f_1), (f_3)–(f_6) hold. Let $(a,b) \in Q^*_{k+1}$, if (a,b) is below $C^{(1)}_{k+1}$ or above $C^{(2)}_{k+1}$, then (1.1) admits at least four nontrivial solutions, including a positive solution u_1, a negative solution u_2 and a sign-changing solution. Moreover, if u_1, u_2 are isolated solutions, then we have $C_q(J,u_i) \cong \delta_{q1} G$, $i = 1, 2$.*

Remark 1.2. Let V be a Kato–Rellich potential. In the case $f(x,t) \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, we just need to assume that (f_1) and (f_6) hold, and $a_0 = b_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_l < a = b < \widetilde{\theta}_l = \Gamma(\mu_{d_l+1}, \sigma_0)$, $\sigma_{\text{dis}}(-\Delta + V) \cap [\lambda_1, \lambda_l] = \{\lambda_i\}_{i=1}^l$, $l \geq 2$, then (1.1) has at least four nontrivial solutions. It is new for (1.1) even in this case.

There are some literatures concerning the study of multiplicity of solutions for (1.1) (see [4], [25], [27], and references therein). In the journey of finding three solutions, we distinguish two solutions by their signs, and hence get a positive solution u_1, a negative solution u_2. In [26], the authors obtained a solution other than u_1 and u_2, denoted by u_3. The sign change of u_3 was also verified by [26]. These are not yet enough to guarantee the existence of the fourth nontrivial solution. It clearly emerges that aiming at four-solution task, we have to resort to Morse theory and distinguish critical points by their critical groups. Further analysis earnestly anticipates an elaborate description for local behavior of each critical point. Note that linking methods via homotopy minimax principle just furnish the information on homotopy groups, so it is indispensable for us to initiate an argument attacking above problem by computing critical groups, and the method is largely different from [8] and [24]. Notice that (1.1) possesses a jumping nonlinear term $f(x,t)$ at 0 and accordingly $J \notin C^2(H^1(\mathbb{R}^N), \mathbb{R})$, so Morse theory of weak smooth functionals on $H^1(\mathbb{R}^N)$ is urgently desired. Corresponding splitting theorem and shifting theorem for a bounded domain $\Omega \subset \mathbb{R}^N$ had been well established in [20] (see also [21] for more general case) and we will give a brief introduction for such contents in Section 7.

The proof of four solutions theorem is arranged in Section 8. We would like to state here some preparation work adapted to our needs. We obtain three critical points of J on $H^1(\mathbb{R}^N)$ and supply precise information on critical groups. Moreover, in Section 8 we build up a weak maximum principle for $\mathbb{R}^N$. Employing such maximum principle and also by Brezis–Martin theorem (see [6] and [28]), we verify that the positive and negative cone of Hilbert space $H^1_m(\mathbb{R}^N)$, denoted by $+P$ and $-P$ respectively, are invariant sets under the negative gradient flow of J on $H^1_m(\mathbb{R}^N)$, and we also show that $((+P)^\delta \cap \mathrm{N}) \cap ((-P)^\delta \cap \mathrm{N}) = \varnothing$ for $\delta > 0$ suitably small, $(\pm P)^\delta := \left\{u \in H^1_m(\mathbb{R}^N) : \inf_{v \in \pm P} \|u - v\|_m \leq \delta\right\}$. Concerning the definitions of $H^1_m(\mathbb{R}^N)$ and $\|\cdot\|_m$, the readers may consult Section 3 for details. Moreover, $\eta(t,u) \subset ((\pm P)^\delta)^\circ$ as $u \in (\pm P)^\delta$, for $\forall t \in (0, +\infty)$, where $((+P)^\delta)^\circ$ and $((-P)^\delta)^\circ$ denote the interior of $(+P)^\delta$ and $(-P)^\delta$ respectively, and η is the negative gradient flow of J on $H^1_m(\mathbb{R}^N)$. Based on minimax argument, we can find two mountain pass type critical points u_1 and u_2 above, s.t., $u_1 \in +P \cap \mathrm{N}$, $u_2 \in -P \cap \mathrm{N}$. As u_3 is sign-changing, combining computations of critical groups of J at infinity and zero with topological analysis of level sets of J, and also using Morse theory of weak smooth functionals on $H^1(\mathbb{R}^N)$, we consequently derive the fourth nontrivial solution.

2. Preliminaries from the spectral theory of Schrödinger operators

2.1. A review of some classical results

We first take a look back at some definitions given by [16]:

Definition 2.1. (see page 136, [16]) Let $V(x) \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$ and be real, with $p = 2$ if $N \le 3$, $p > 2$ if $N = 4$ and $p > \frac{N}{2}$ if $N \ge 5$, then V is called Kato–Rellich (K–R) potential.

Definition 2.2. (see Definition 14.7, [16]) A potential function $V(x)$ is called a Kato potential if V is real and $V \in L^2(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)_\epsilon$, where the ϵ indicates that for any $\epsilon > 0$, we can decompose $V = V_1 + V_2$ with $V_1 \in L^2(\mathbb{R}^N)$ and $V_2 \in L^\infty(\mathbb{R}^N)$, with $\|V_2\|_{L^\infty} < \epsilon$.

Proposition 2.3. *(see Corollary 14.10, [16]) If V is a real Kato potential, then* $\sigma_{\text{ess}}(-\Delta + V) = [0, +\infty)$.

For the readers' convenience, we present a proof for the following well-known result (see [16]):

Proposition 2.4. *Let V be a real K–R potential, $\lim_{|x| \to +\infty} V(x) = 0$, then* $\sigma_{\text{ess}}(-\Delta + V) = [0, +\infty)$.

Proof. In view of the hypothesis, $\forall \varepsilon > 0$, $\exists R > 0$, for $x \in \mathbb{R}^N$, $|x| \ge R$, $|V(x)| \le \varepsilon$. Write $V = V_1 + V_2$, $V_1 \in L^p(\mathbb{R}^N)$, $V_2 \in L^\infty(\mathbb{R}^N)$. Notice that

$$\begin{aligned} V &= \chi_{B(0,R)} V_1 + \chi_{B^c(0,R)} V_1 + \chi_{B(0,R)} V_2 + \chi_{B^c(0,R)} V_2 \\ &= V_1^{(R)} + V_2^{(R)}, \end{aligned} \tag{2.1}$$

where $V_1^{(R)} = \chi_{B(0,R)} V$, $V_2^{(R)} = \chi_{B^c(0,R)} V$, it follows that for $x \in \mathbb{R}^N$, $|x| \ge R$, $\left|V_2^{(R)}(x)\right| = |V(x)| \le \varepsilon$. Observe that $V_1^{(R)} \in L^p(\mathbb{R}^N) \Rightarrow V_1^{(R)} \in L^2(\mathbb{R}^N)$, so V is a Kato potential. Using Proposition 2.3 we arrive at the conclusion. □

Corollary 2.5. *Let V be a real K–R potential, $\lim_{|x| \to +\infty} V(x) = \alpha$, then* $\sigma_{\text{ess}}(-\Delta + V) = [\alpha, +\infty)$.

As is known, if the real potential V is in the K–R class, then $-\Delta + V$ is a self-adjoint, semibounded Schrödinger operator with domain $H^2(\mathbb{R}^N)$ and there is a simplified version of Persson's theorem which proves that the behavior of the potential V at infinity determines $\sigma_{\text{ess}}(-\Delta + V)$:

Proposition 2.6. *(see Theorem 14.11, [16]) Let V be a real-valued potential in the K–R class, and let $-\Delta + V$ be the corresponding self-adjoint, semibounded Schrödinger operator with domain $H^2(\mathbb{R}^N)$. Then, the bottom of the essential spectrum is given by*

$$\inf \sigma_{\text{ess}}(-\Delta + V) = \sup_{\kappa \subset \mathbb{R}^N} \inf_{\substack{\phi \in C_0^\infty\left(\mathbb{R}^N \setminus \kappa\right) \\ \phi \neq 0}} \frac{\langle(-\Delta + V)\phi, \phi\rangle_{L^2(\mathbb{R}^N)}}{\|\phi\|^2_{L^2(\mathbb{R}^N)}}, \tag{2.2}$$

where the supremum is over all compact subsets $\kappa \subset \mathbb{R}^N$.

Suppose:

(V_1) $V(x)$ is a real potential, and $\exists F(x) \in L^\infty\left(\mathbb{R}^N\right)$, $\alpha = \inf_{x \in \mathbb{R}^N} F(x) \in (-\infty, +\infty)$, s.t. $\lim_{|x| \to \infty} (V(x) - F(x)) = 0$.

As $\lim_{|x| \to \infty} V(x) = \alpha$, by taking $F(x) \equiv \alpha$, the hypothesis (V_1) evidently follows. With the aid of Proposition 2.4, a general version of Corollary 2.5 can be derived:

Proposition 2.7. *Let $V(x)$ be a real K–R potential and suppose (V_1), then*

$$\sigma_{\text{ess}}(-\Delta + V) \subset [\alpha, +\infty). \tag{2.3}$$

Proof. Based on the assumption, decompose $V = V_1 + V_2$, $V_1 \in L^p\left(\mathbb{R}^N\right)$, $V_2 \in L^\infty\left(\mathbb{R}^N\right)$. Set $\widetilde{V} = V_1 + \widetilde{V}_2$, $\widetilde{V}_2 = V_2 - F(x)$. Employing Proposition 2.4 we obtain

$$\sigma_{\text{ess}}\left(-\Delta + \widetilde{V}\right) = [0, +\infty). \tag{2.4}$$

By (2.2) we have

$$\begin{aligned} 0 &= \inf \sigma_{\text{ess}}\left(-\Delta + \widetilde{V}\right) \\ &= \sup_{\kappa \subset \mathbb{R}^N} \inf_{\phi \in C_0^\infty\left(\mathbb{R}^N \setminus \kappa\right) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla \phi|^2 + \widetilde{V}\phi^2}{\|\phi\|^2_{L^2(\mathbb{R}^N)}} \\ &\leq \sup_{\kappa \subset \mathbb{R}^N} \inf_{\phi \in C_0^\infty\left(\mathbb{R}^N \setminus \kappa\right) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla \phi|^2 + V\phi^2}{\|\phi\|^2_{L^2(\mathbb{R}^N)}} - \alpha, \end{aligned} \tag{2.5}$$

ending the proof. □

2.2. A min–max principle on $H^1\left(\mathbb{R}^N\right)$

Proposition 2.8. *(min–max principle, operator form) (see Theorem XIII.1, [33]) Let H be a self-adjoint operator on Hilbert space E and H is bounded from below, i.e., $H \geq cI$ for some c. Define $\mu_k(H) = \sup_{\mathfrak{B} = span\left\{\psi_1, \cdots, \psi_{k-1}\right\}, \psi_i \in D(H)} \inf_{\substack{\psi \in D(H), \|\psi\|_E = 1 \\ \psi \in \mathfrak{B}^\perp}} \langle H\psi, \psi\rangle_E$. Then, for each fixed k, either:*

(a) there are k eigenvalues (counting degenerate eigenvalues a number of times equal to their multiplicity) below the bottom of the essential spectrum, and $\mu_k(H)$ is the k-th eigenvalue counting multiplicity; or (b) μ_k is the bottom of the essential spectrum, i.e., $\mu_k = \inf\{\lambda : \lambda \in \sigma_{\text{ess}}(H)\}$ and in that case $\mu_k = \mu_{k+1} = \mu_{k+2} = \cdots$ and there are at most $k - 1$ eigenvalues (counting multiplicity) below μ_k.

Lemma 2.9. *Suppose $V(x) \in L^p(\mathbb{R}^N)$, $p = 2$ if $N \le 3$, $p > 2$ if $N = 4$, $p > \frac{N}{2}$ if $N \ge 5$, or $p = +\infty$, and define*

$$\eta_k := \inf_{\substack{\mathfrak{B} \subset H^1(\mathbb{R}^N) \\ \dim \mathfrak{B} = k}} \max_{u \in \mathfrak{B} \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x) u^2}{\int_{\mathbb{R}^N} u^2}; \tag{2.6}$$

$$\sigma_k := \inf_{\substack{\mathfrak{B} \subset H^2(\mathbb{R}^N) \\ \dim \mathfrak{B} = k}} \max_{u \in \mathfrak{B} \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x) u^2}{\int_{\mathbb{R}^N} u^2}, \tag{2.7}$$

then $\eta_k = \sigma_k$ for $\forall k \in \mathbb{N}$.

Proof. One deduces from [40] that $-\Delta + V$ is bounded from below on $H^2(\mathbb{R}^N)$ for K–R potential and so $\eta_k > -\infty$. Choose linearly independent $\omega_1, \cdots, \omega_k \in H^1(\mathbb{R}^N)$ and set $\mathfrak{B} = \operatorname{span}\{\omega_1, \cdots, \omega_k\}$. As $C_0^\infty(\mathbb{R}^N)$ is dense in $H^1(\mathbb{R}^N)$, for ω_i we can find $\left\{\omega_i^{(m)}\right\}_{m=1}^{+\infty} \subset C_0^\infty(\mathbb{R}^N)$ such that $\omega_i^{(m)} \to \omega_i$ in $H^1(\mathbb{R}^N)$ as $m \to +\infty$, $i = 1, 2, \cdots, k$.

Step one: We claim that $\exists M \in \mathbb{N}$, $\forall m \ge M$, $m \in \mathbb{N}$, $\omega_1^{(m)}, \cdots, \omega_k^{(m)}$ is linearly independent. By way of negation, $\exists j_0 \in \mathbb{N}$, and a sequence $\{m_l\}_{l=1}^{+\infty} \in \mathbb{N}$, $m_l \to +\infty$ as $l \to +\infty$, s.t. $\omega_{j_0}^{(m_l)} = \sum_{i \ne j_0} \alpha_i^{(m_l)} \omega_i^{(m_l)}$.

Set $\left|\alpha^{(m_l)}\right| = \left(\sum_{i \ne j_0} \left|\alpha_i^{(m_l)}\right|^2\right)^{\frac{1}{2}}$ and $A_{m_l}^{(i)} = \frac{\alpha_i^{(m_l)}}{\left|\alpha^{(m_l)}\right|}$, $i \ne j_0$. We are confronted with two cases:
(1) $\left|\alpha^{(m_l)}\right| \to +\infty$, $l \to +\infty$.
Then we have

$$\frac{\omega_{j_0}^{(m_l)}}{\left|\alpha^{(m_l)}\right|} = \sum_{i \ne j_0} A_{m_l}^{(i)} \omega_i^{(m_l)} \tag{2.8}$$

and infer that $\exists M^* > 0$, $\exists L \in \mathbb{N}$, $\forall l \ge L$, $\exists i^{(l)} \ne j_0$ such that

$$\left|A_{m_l}^{(i^{(l)})}\right| \ge M^*. \tag{2.9}$$

Otherwise, $\forall \varepsilon > 0$, $\forall \widetilde{L} \in \mathbb{N}$, $\exists \widetilde{l} \ge \widetilde{L}$,

$$\left|A_{m_{\widetilde{l}}}^{(i)}\right| < \varepsilon, i \ne j_0. \tag{2.10}$$

(2.10) yields

$$1 = \sum_{i \ne j_0} \left|A_{m_{\widetilde{l}}}^{(i)}\right|^2 < (k-1)\varepsilon^2, \tag{2.11}$$

which is impossible as ε is small enough.

Hence, for $l \geq L$, up to a rewritten subsequence $\left\{A_{m_l}^{(i^{(l)})}\right\}_{l=L}^{+\infty}$, $\exists i_0 \neq j_0$, $i^{(l)} = i_0$. Since $\left|A_{m_l}^{(i)}\right| \leq 1$, there exist A_i and also a subsequence, still labeled by $\left\{A_{m_l}^{(i)}\right\}_{l=1}^{+\infty}$ such that $\lim_{l\to+\infty} A_{m_l}^{(i)} = A_i$. Take limit for (2.8) in $H^1\left(\mathbb{R}^N\right)$, we obtain

$$\sum_{i\neq j_0} A_i\omega_i = 0. \tag{2.12}$$

Notice that $\left|A_{i_0}\right| \geq M^* > 0$, so $\omega_1, \cdots, \omega_{j_0-1}, \omega_{j_0+1}, \cdots, \omega_k$ are linearly dependent which contradicts $\dim \mathfrak{B} = k$.

(2) $\left\{\alpha^{(m_l)}\right\}_{l=1}^{+\infty}$ is bounded.

The assertion is obvious in this case.

Step two. Define

$$\widetilde{\lambda} := \max_{u\in\mathfrak{B}\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)\,u^2}{\int_{\mathbb{R}^N} u^2};$$

$$\widetilde{\lambda}_m := \max_{u\in\mathfrak{B}^{(m)}\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)\,u^2}{\int_{\mathbb{R}^N} u^2};$$

$$\mathfrak{B}^{(m)} := \operatorname{span}\left\{\omega_1^{(m)}, \cdots, \omega_k^{(m)}\right\}, m \text{ large enough.}$$

Now we show that $\widetilde{\lambda}_m \to \widetilde{\lambda}$. To see this, we assume that $\widetilde{\lambda}_m$ is achieved by $\widetilde{u}_m \in \mathfrak{B}^{(m)}$, $\|\widetilde{u}_m\|_{H^1(\mathbb{R}^N)} = 1$. Set $\widetilde{u}_m = \sum_{i=1}^{k} \widetilde{\alpha}_i^{(m)}\omega_i^{(m)}$, by the proof of step one we know that there exists $M_0 > 0$, $\left|\widetilde{\alpha}_i^{(m)}\right| \leq M_0$. Hence we can find a renamed subsequence $\left\{\widetilde{\alpha}_i^{(m)}\right\}_{m=1}^{+\infty}$, $\lim_{m\to+\infty} \widetilde{\alpha}_i^{(m)} = \widetilde{\alpha}_i$. So $\widetilde{u}_m \to \widetilde{u}_0 = \sum_{i=1}^{k} \widetilde{\alpha}_i\omega_i$ in $H^1\left(\mathbb{R}^N\right)$ and thus we obtain

$$\widetilde{\lambda}_m \to \widetilde{\lambda}_0 := \frac{\int_{\mathbb{R}^N} |\nabla \widetilde{u}_0|^2 + V(x)\,\widetilde{u}_0^2}{\int_{\mathbb{R}^N} \widetilde{u}_0^2} \leq \widetilde{\lambda}. \tag{2.13}$$

On the other hand, we assume that $\widetilde{\lambda}$ is achieved by $u_0 = \sum_{i=1}^{k} \alpha_i\omega_i$ and set $\|u_0\| = 1$, then there exist $\{u_m\}_{m=1}^{+\infty} \subset C_0^\infty\left(\mathbb{R}^N\right)$, $u_m = \sum_{i=1}^{k} \alpha_i^*\omega_i^{(m)} \to u_0$ in $H^1\left(\mathbb{R}^N\right)$. If $p = +\infty$, the conclusion is evident so we just need to deal with the case $p > \frac{N}{2}$. Using embedding theorem we get

$$u_m \to u_0 \text{ in } L^q\left(\mathbb{R}^N\right), \tag{2.14}$$

for $q \in \left[2, 2^*\right]$, if $N \geq 3$ and for $q \in [2, +\infty)$ if $N = 1, 2$.

We claim that

$$\lim_{m\to+\infty}\int_{\mathbb{R}^N} V(x)\,u_m^2=\int_{\mathbb{R}^N} V(x)\,u_0^2 \tag{2.15}$$

Divide the proof into two cases:

(i) $N\ge 4$. Since $p>\frac{N}{2}$, $\frac{1}{p}+\frac{1}{q}=1\Rightarrow q\in\left(1,\frac{N}{N-2}\right)$. By Hölder inequality and Cauchy–Schwarz inequality,

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} V(x)\left(u_m^2-u_0^2\right)\right|\\
&\le \|V\|_{L^p}\cdot\left(\int_{\mathbb{R}^N}|u_m+u_0|^q\cdot|(u_m-u_0)|^q\,dx\right)^{\frac{1}{q}}\\
&\le \|V\|_{L^p}\cdot\left(\|u_m\|_{L^{2q}}+\|u_0\|_{L^{2q}}\right)\cdot\|u_m-u_0\|_{L^{2q}}\\
&\le C\,\|V\|_{L^p}\cdot(\|u_m\|+\|u_0\|)\cdot\|u_m-u_0\|\to 0.
\end{aligned} \tag{2.16}$$

(ii) $N\le 3$. Observe that $V\in L^2\left(\mathbb{R}^N\right)$ if $N\le 3$, taking $q=2$ shows

$$\left|\int_{\mathbb{R}^N} V(x)\left(u_m^2-u_0^2\right)\right|\le C\,\|V\|_{L^2}\cdot(\|u_m\|+\|u_0\|)\cdot\|u_m-u_0\|\to 0. \tag{2.17}$$

As $u_m\to u_0$ in L^2, we have

$$\|u_m\|_{L^2}\to\|u_0\|_{L^2}\,. \tag{2.18}$$

Notice that

$$\begin{aligned}
&\langle\nabla u_m,\nabla u_m\rangle_{L^2}-\langle\nabla u_0,\nabla u_0\rangle_{L^2}\\
&=\int_{\mathbb{R}^N}|\nabla(u_m-u_0)|^2+2\int_{\mathbb{R}^N}\nabla(u_m-u_0)\,\nabla u_0\to 0,
\end{aligned} \tag{2.19}$$

consequently,

$$\widetilde{\lambda}\leftarrow\frac{\int_{\mathbb{R}^N}|\nabla u_m|^2+V(x)\,u_m^2}{\int_{\mathbb{R}^N}u_m^2}\le\widetilde{\lambda}_m. \tag{2.20}$$

Combining (2.13) with (2.20) we get $\widetilde{\lambda}_0=\widetilde{\lambda}$.

Step three: Define $\lambda:=\inf\limits_{\substack{\mathfrak{B}\subset H^1\left(\mathbb{R}^N\right)\\ \dim\mathfrak{B}=k}}\max\limits_{u\in\mathfrak{B}\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla u|^2+V(x)u^2}{\int_{\mathbb{R}^N}u^2}$, then for $\forall\varepsilon>0$, $\exists\mathfrak{B}\subset H^1\left(\mathbb{R}^N\right)$,

$$\max_{u\in\mathfrak{B}\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2}{\int_{\mathbb{R}^N} u^2} < \lambda + \frac{\varepsilon}{2}. \tag{2.21}$$

Step two indicates that there exists $\widehat{\mathfrak{B}} \subset C_0^\infty(\mathbb{R}^N)$, $\dim\widehat{\mathfrak{B}} = k$, s.t.

$$\left|\max_{u\in\mathfrak{B}\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2}{\int_{\mathbb{R}^N} u^2} - \max_{u\in\widehat{\mathfrak{B}}\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2}{\int_{\mathbb{R}^N} u^2}\right| < \frac{\varepsilon}{2}. \tag{2.22}$$

Therefore, by combining (2.21) with (2.22) we have

$$\max_{u\in\widehat{\mathfrak{B}}\backslash\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2}{\int_{\mathbb{R}^N} u^2} < \lambda + \varepsilon. \tag{2.23}$$

The proof is complete. □

Corollary 2.10. *Suppose that V is a real K–R potential, then $\eta_k = \sigma_k$ for $\forall k \in \mathbb{N}$..*

Set $A = -\Delta + V$ and define

$$\mu_k := \sup_{\substack{\mathfrak{B}_{k-1}\subset H^2\\ \dim\mathfrak{B}_{k-1}=k-1}} \inf_{\substack{\psi\in\mathfrak{B}_{k-1}^\perp\cap H^2\\ \|\psi\|_{L^2}=1}} \langle A\psi,\psi\rangle_{L^2}, \tag{2.24}$$

$$\alpha_k := \sup_{\substack{\mathfrak{B}_{k-1}\subset H^1\\ \dim\mathfrak{B}_{k-1}=k-1}} \inf_{\substack{\psi\in\mathfrak{B}_{k-1}^\perp\cap H^1\\ \|\psi\|_{L^2}=1}} \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(x)\psi^2, \tag{2.25}$$

$$\beta_k := \inf_{\substack{\mathfrak{B}_k\subset H^2\\ \dim\mathfrak{B}_k=k}} \max_{\substack{\psi\in\mathfrak{B}_k\\ \|\psi\|_{L^2}=1}} \langle A\psi,\psi\rangle_{L^2}, \tag{2.26}$$

$$\gamma_k := \inf_{\substack{\mathfrak{B}_k\subset H^1\\ \dim\mathfrak{B}_k=k}} \max_{\substack{\psi\in\mathfrak{B}_k\\ \|\psi\|_{L^2}=1}} \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(x)\psi^2, \tag{2.27}$$

where $\mathfrak{B}_{k-1}^\perp$ is the orthogonal complement in $L^2(\mathbb{R}^N)$ of $\mathfrak{B}_{k-1}$. Obviously, $\beta_k = \sigma_k = \eta_k = \gamma_k$ for $\forall k \in \mathbb{N}$.

Let V be a K–R potential, $\sigma_{\text{dis}}(A) \neq \varnothing$, $\inf\sigma(A) = \inf\sigma_{\text{dis}}(A)$, $\{\lambda_i\}_{i=1}^l \in \sigma_{\text{dis}}(A)$, $l \geq 1$, s.t., $\lambda_1 = \inf\sigma(A) < \lambda_2 < \cdots < \lambda_l < \sigma_0$, $\sigma_{\text{dis}}(A) \cap [\lambda_1, \sigma_0) = \{\lambda_i\}_{i=1}^l$, $\sigma_0 := \inf\sigma_{\text{ess}}(A)$. Let N_l be the subspace of $D = D(A) = H^2(\mathbb{R}^N)$ spanned by the eigenfunctions $\psi_1, \cdots, \psi_{d_l}$ corresponding to $\lambda_1, \cdots, \lambda_l$, $d_l = \dim N_l$. Then $H^1(\mathbb{R}^N) = N_l \oplus M_l$, where $\oplus$ denotes decomposition into direct sum in $L^2(\mathbb{R}^N)$, $M_l := N_l^\perp \cap H^1(\mathbb{R}^N)$. Let $E_{k-1} := \text{span}\{\psi_1, \cdots \psi_{k-1}\}$, $k \leq d_l + 1$, and set

$$\zeta_k := \inf_{\substack{\psi\in E_{k-1}^\perp\cap H^2\\ \|\psi\|_{L^2}=1}} \langle A\psi,\psi\rangle_{L^2}, \tag{2.28}$$

$$\xi_k := \inf_{\substack{\psi \in E_{k-1}^{\perp} \cap H^1 \\ \|\psi\|_{L^2}=1}} \int_{\mathbb{R}^N} |\nabla \psi|^2 + V(x)\psi^2. \tag{2.29}$$

Under the above hypotheses, our consequence concerning min–max principle reads:

Lemma 2.11. *(i)* $\xi_k = \mu_k$ *for* $k \le d_l + 1$*; (ii)* $\mu_k = \alpha_k = \gamma_k$, $\forall k \in \mathbb{N}$.

Proof. Divide the proof into two cases:

(1) $k \le d_l + 1$.

We first claim $\mu_k = \zeta_k$. Obviously, $\zeta_k \le \mu_k$, so we focus on $\mu_k \le \zeta_k$. For $\forall \mathfrak{B}_{k-1} \subset H^2$, take two cases into account:

(a) $\exists \psi_0 \in \mathfrak{B}_{k-1}^{\perp} \cap E_{k-1} \setminus \{0\}$. Set $\|\psi_0\|_{L^2} = 1$. Denote by E_λ the spectral system of A. Let $P_{E(\mu,+\infty)}$ be the orthogonal projection operator $\chi_{(\mu,+\infty)}$, where $\chi_{(\mu,+\infty)}$ is the characteristic function of $(\mu, +\infty)$, we have $\int_{\mu}^{+\infty} dE_\lambda = P_{E(\mu,+\infty)}$, $A = \int_{-\infty}^{+\infty} \lambda dE_\lambda$.

Due to the fact

$$\begin{aligned} \inf_{\substack{\psi \in E_{k-1}^{\perp} \cap H^2 \\ \|\psi\|_{L^2}=1}} \langle A\psi, \psi\rangle_{L^2} &= \inf_{\substack{\psi \in E_{k-1}^{\perp} \cap H^2 \\ \|\psi\|_{L^2}=1}} \int_{-\infty}^{+\infty} \lambda d\langle E_\lambda \psi, \psi\rangle_{L^2} \\ &= \inf_{\substack{\psi \in E_{k-1}^{\perp} \cap H^2 \\ \|\psi\|_{L^2}=1}} \int_{\mu_{k-1}}^{+\infty} \lambda d\langle E_\lambda \psi, \psi\rangle_{L^2} \ge \mu_{k-1}, \end{aligned} \tag{2.30}$$

hence,

$$\inf_{\substack{\psi \in \mathfrak{B}_{k-1}^{\perp} \cap H^2 \\ \|\psi\|_{L^2}=1}} \langle A\psi, \psi\rangle_{L^2} \le \langle A\psi_0, \psi_0\rangle_{L^2} \le \mu_{k-1} \le \zeta_k. \tag{2.31}$$

(b) $\mathfrak{B}_{k-1}^{\perp} \cap E_{k-1} = \{0\}$. Observe that the map $P_{E_{k-1}^{\perp}} : \mathfrak{B}_{k-1}^{\perp} \cap H^2 \to E_{k-1}^{\perp} \cap H^2$ is one to one onto. Actually, we just need to show the surjection since the injection is evident. Otherwise, $\exists y \in E_{k-1}^{\perp} \cap H^2 \setminus \{0\}$, for $\forall x \in \mathfrak{B}_{k-1}^{\perp} \cap H^2$, $P_{E_{k-1}^{\perp}} x \ne y$. Set $\|y\|_{L^2} = 1$, $F_k = \operatorname{span}\{y, \psi_1, \cdots, \psi_{k-1}\}$. As $\dim F_k = k > \dim \mathfrak{B}_{k-1} = k-1$, there exists $z \in F_k \cap \mathfrak{B}_{k-1}^{\perp} \setminus \{0\}$. Denote $z = \alpha y + \sum_{i=1}^{k-1} \beta_i \psi_i$. Clearly, $\alpha \ne 0$. Then $P_{E_{k-1}^{\perp}} z = \alpha y$, i.e., $P_{E_{k-1}^{\perp}} \frac{z}{\alpha} = y$, violating the hypothesis. Consequently, for $\forall \varphi \in E_{k-1}^{\perp} \cap H^2$, $\exists \psi \in \mathfrak{B}_{k-1}^{\perp} \cap H^2$, $P_{E_{k-1}^{\perp}} \psi = \varphi$. Take the minimizing sequence $\{\varphi_n\}_{n=1}^{+\infty}$ of (2.28), $\varphi_n \in E_{k-1}^{\perp} \cap H^2$, $\|\varphi_n\|_{L^2} = 1$, s.t., $\varepsilon_n > 0$ and

$$\langle A\varphi_n, \varphi_n\rangle_{L^2} \le \zeta_k + \varepsilon_n. \tag{2.32}$$

For $\forall \psi \in \mathfrak{B}_{k-1}^{\perp} \cap H^2$, $\|\psi\|_{L^2} = 1$, set $\psi = P_{E_{k-1}^{\perp}} \psi + P_{E_{k-1}} \psi$. For above φ_n, $\exists s_n \in (0, 1]$, $\psi_n^* \in \mathfrak{B}_{k-1}^{\perp} \cap H^2$, $\|\psi_n^*\|_{L^2} = 1$, s.t.,

$$\psi_n^* = P_{E_{k-1}^{\perp}} \psi_n^* + P_{E_{k-1}} \psi_n^* = s_n \varphi_n + P_{E_{k-1}} \psi_n^*. \tag{2.33}$$

Set $P_{E_{k-1}}\psi_n^* = t_n\phi_n$, $t_n^2 = 1 - s_n^2$, $\|\phi_n\|_{L^2} = 1$. Therefore,

$$\begin{aligned}\inf_{\substack{\psi\in\mathfrak{B}_{k-1}^{\perp}\cap H^2\\ \|\psi\|_{L^2}=1}}\langle A\psi,\psi\rangle_{L^2} &\le \left\langle A\psi_n^*,\psi_n^*\right\rangle_{L^2}\\ &\le (\zeta_k+\varepsilon_n)\,s_n^2 + \mu_{k-1}t_n^2\\ &\le \zeta_k+\varepsilon_n \end{aligned} \tag{2.34}$$

and this yields

$$\inf_{\substack{\psi\in\mathfrak{B}_{k-1}^{\perp}\cap H^2\\ \|\psi\|_{L^2}=1}}\langle A\psi,\psi\rangle_{L^2} \le \zeta_k. \tag{2.35}$$

(2.31) together with (2.35) derives

$$\sup_{\substack{\mathfrak{B}_{k-1}\subset H^2\\ \dim\mathfrak{B}_{k-1}=k-1}}\ \inf_{\substack{\psi\in\mathfrak{B}_{k-1}^{\perp}\cap H^2\\ \|\psi\|_{L^2}=1}}\langle A\psi,\psi\rangle_{L^2} \le \zeta_k. \tag{2.36}$$

We conclude the claim as predicted.

Notice that $\zeta_k = \xi_k$ for $k \le d_l + 1$ (see Lemma 9.1 for more detail), so $\mu_k = \xi_k$ for $k \le d_l + 1$. That's the precise statement (i).

Next we show $\zeta_k = \beta_k$ for $k \le d_l + 1$. Note that $\forall\mathfrak{B}_k \subset H^2$, $\dim\mathfrak{B}_k = k$, $\exists\psi^* \in \mathfrak{B}_k \cap E_{k-1}^{\perp}$, $\|\psi^*\|_{L^2} = 1$, then we obtain

$$\max_{\substack{\psi\in\mathfrak{B}_k\\ \|\psi\|_{L^2}=1}}\langle A\psi,\psi\rangle_{L^2} \ge \left\langle A\psi^*,\psi^*\right\rangle_{L^2} \ge \zeta_k \tag{2.37}$$

alluding to $\beta_k \ge \zeta_k$ for $k \le d_l + 1$. On the other side, for $k \le d_l$,

$$\beta_k \le \max_{\substack{\psi\in E_k\\ \|\psi\|_{L^2}=1}}\langle A\psi,\psi\rangle_{L^2} \le \mu_k = \zeta_k, \tag{2.38}$$

and as $k = d_l + 1$, for above φ_n, set $E_{d_l+1}^{(n)} = \operatorname{span}\left\{\psi_1,\cdots,\psi_{d_l},\varphi_n\right\}$, hence,

$$\beta_{d_l+1} \le \max_{\substack{\psi\in E_{d_l+1}^{(n)}\\ \|\psi\|_{L^2}=1}}\langle A\psi,\psi\rangle_{L^2} \le \langle A\varphi_n,\varphi_n\rangle_{L^2} \le \zeta_{d_l+1}+\varepsilon_n, \tag{2.39}$$

and this gets $\beta_{d_l+1} \le \zeta_{d_l+1}$. The combination of (2.37), (2.38) and (2.39) ends the proof.

It is left to us to verify $\alpha_k = \xi_k$ for $k \le d_l + 1$. For $\forall\mathfrak{B}_{k-1} \subset H^1$, take two cases into account: $\langle 1\rangle$ $\exists\psi_0 \in \mathfrak{B}_{k-1}^{\perp} \cap E_{k-1}\backslash\{0\}$. Set $\|\psi_0\|_{L^2} = 1$. As $\zeta_k = \xi_k$ for $k \le d_l + 1$, we have

$$\begin{aligned}
&\inf_{\substack{\psi\in\mathfrak{B}_{k-1}^{\perp}\cap H^1\\ \|\psi\|_{L^2}=1}} \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(x)\,\psi^2 \\
&\le \inf_{\substack{\psi\in\mathfrak{B}_{k-1}^{\perp}\cap E_{k-1}\\ \|\psi\|_{L^2}=1}} \langle A\psi,\psi\rangle_{L^2} \\
&\le \langle A\psi_0,\psi_0\rangle_{L^2} \le \mu_{k-1} \le \zeta_k = \xi_k.
\end{aligned} \tag{2.40}$$

$\langle 2\rangle$ $\mathfrak{B}_{k-1}^{\perp}\cap E_{k-1}=\{0\}$. The argument is quite similar to (b).

$\langle 1\rangle + \langle 2\rangle \Rightarrow \alpha_k \le \xi_k$.

On the other hand, on account of the definition, $\alpha_k \ge \xi_k$. The assertion follows.

Overall, we conclude the proof of (ii) for $k \le d_l + 1$ with the help of Corollary 2.10.

(2) $k > d_l + 1$.

We can always choose $\{\psi_i\}_{i=d_l+1}^{+\infty} \in E_{d_l}^{\perp} \cap H^2$ such that $\{\psi_i\}_{i=1}^{+\infty}$ is a complete basis of L^2, $\langle \psi_i, \psi_j\rangle_{L^2} = 0$, for $i \ne j$, $i, j \ge d_l + 1$.

Observe that $\forall\varepsilon > 0$, $\forall\mathfrak{B}_{k-1} \subset H^1$, $\dim\mathfrak{B}_{k-1} = k-1$, $\mathfrak{B}_{k-1}^{\perp} \cap \left(E_{\sigma_0+\varepsilon} - E_{\sigma_0}\right) H^2 \ne \{0\}$. This is due to $\dim\left(E_{\sigma_0+\varepsilon} - E_{\sigma_0}\right) H^2 = \infty$.

For fixed $\varepsilon_n > 0$, then there exists $\mathfrak{B}_{k-1}^{(n)} \subset H^1$, $\dim\mathfrak{B}_{k-1}^{(n)} = k-1$, and $\widetilde{\psi}_n \in \left(\mathfrak{B}_{k-1}^{(n)}\right)^{\perp} \cap \left(E_{\sigma_0+\varepsilon_n} - E_{\sigma_0}\right) H^2$, $\left\|\widetilde{\psi}_n\right\|_{L^2} = 1$, s.t.,

$$\begin{aligned}
\alpha_k - \varepsilon_n &\le \inf_{\substack{\psi\in\left(\mathfrak{B}_{k-1}^{(n)}\right)^{\perp}\cap H^1\\ \|\psi\|_{L^2}=1}} \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(x)\,\psi^2 \\
&\le \inf_{\substack{\psi\in\left(\mathfrak{B}_{k-1}^{(n)}\right)^{\perp}\cap H^2\\ \|\psi\|_{L^2}=1}} \langle A\psi,\psi\rangle_{L^2} \\
&\le \left\langle A\widetilde{\psi}_n, \widetilde{\psi}_n\right\rangle_{L^2} \le \sigma_0 + \varepsilon_n.
\end{aligned} \tag{2.41}$$

Let $\varepsilon_n \to 0$, (2.41) yields

$$\alpha_k = \sup_{\substack{\mathfrak{B}_{k-1}\subset H^1\\ \dim\mathfrak{B}_{k-1}=k-1}} \inf_{\substack{\psi\in\mathfrak{B}_{k-1}^{\perp}\cap H^1\\ \|\psi\|_{L^2}=1}} \int_{\mathbb{R}^N} |\nabla\psi|^2 + V(x)\,\psi^2 \le \sigma_0 = \mu_k. \tag{2.42}$$

Due to the fact $\sigma_0 = \mu_{d_l+1} = \alpha_{d_l+1} \le \alpha_k$ for $k \ge d_l + 2$, we get $\alpha_k = \mu_k$ for $k \ge d_l + 2$.

We now remain to prove $\mu_k = \beta_k$ for $\forall k \in \mathbb{N}$. In view of the definitions of β_k and μ_k, for $\varepsilon_n > 0$, there exists $\mathfrak{B}_k^{*(n)} \subset H^2$, $\dim\mathfrak{B}_k^{*(n)} = k$, and $\widetilde{\mathfrak{B}}_{k-1}^{(n)} \subset H^2$, $\dim\widetilde{\mathfrak{B}}_{k-1}^{(n)} = k-1$, s.t.

$$\max_{\substack{\psi\in\mathfrak{B}_k^{*(n)}\\ \|\psi\|_{L^2}=1}} \langle A\psi,\psi\rangle_{L^2} \le \beta_k + \varepsilon_n, \tag{2.43}$$

$$\inf_{\substack{\psi\in\left(\widetilde{\mathfrak{B}}_{k-1}^{(n)}\right)^{\perp}\cap H^2\\ \|\psi\|_{L^2}=1}} \langle A\psi,\psi\rangle_{L^2} \ge \mu_k - \varepsilon_n. \tag{2.44}$$

Since $\dim \mathfrak{B}_k^{*(n)} > \dim \widetilde{\mathfrak{B}}_{k-1}^{(n)}$, there exists $\psi_0^{(n)} \in \mathfrak{B}_k^{*(n)} \cap \left(\widetilde{\mathfrak{B}}_{k-1}^{(n)}\right)^{\perp}$, $\left\|\psi_0^{(n)}\right\|_{L^2} = 1$. Based on this, the combination of (2.43) and (2.44) yields $\beta_k \geq \mu_k$ for $\forall k \in \mathbb{N}$.

The other side of the shield, for $\forall \varepsilon > 0$, notice that $\dim\left(E_{\sigma_0+\varepsilon} - E_{\sigma_0}\right) H^2 = \infty$. If $k \geq d_l + 2$, pick $\left\{\xi_i^{(\varepsilon)}\right\}_{i=d_l+1}^{k}$, s.t., $\sigma_0 < \xi_{d_l+1}^{(\varepsilon)} < \xi_{d_l+2}^{(\varepsilon)} < \cdots < \xi_k^{(\varepsilon)} < \sigma_0 + \varepsilon$, and take $\left\{\psi_i^{(\varepsilon)}\right\}_{i=d_l+1}^{k} \subset \left(E_{\sigma_0+\varepsilon} - E_{\sigma_0}\right) H^2$, s.t., $\left\langle A\psi_i^{(\varepsilon)}, \psi_j^{(\varepsilon)}\right\rangle_{L^2} = 0$ for $i \neq j$, $d_l + 1 \leq i, j \leq k$, $\sigma_0 < \left\langle A\psi_{d_l+1}^{(\varepsilon)}, \psi_{d_l+1}^{(\varepsilon)}\right\rangle_{L^2} < \xi_{d_l+1}^{(\varepsilon)}$, $\xi_{i-1}^{(\varepsilon)} < \left\langle A\psi_i^{(\varepsilon)}, \psi_i^{(\varepsilon)}\right\rangle_{L^2} < \xi_i^{(\varepsilon)}$ for $d_l + 2 \leq i \leq k$, and $\left(E_{\xi_j^{(\varepsilon)}} - E_{\xi_{j-1}^{(\varepsilon)}}\right)\psi_j^{(\varepsilon)} = \psi_j^{(\varepsilon)}$ for $d_l+1 \leq j \leq k$. Set $\widetilde{E}_{k,\varepsilon} = \operatorname{span}\left\{\psi_1, \cdots, \psi_{d_l}, \psi_{d_l+1}^{(\varepsilon)}, \cdots, \psi_k^{(\varepsilon)}\right\}$. Thereby,

$$\beta_k \leq \max_{\substack{\psi \in \widetilde{E}_{k,\varepsilon} \\ \|\psi\|_{L^2}=1}} \langle A\psi, \psi\rangle_{L^2} < \sigma_0 + \varepsilon, \tag{2.45}$$

alluding to $\beta_k \leq \sigma_0 = \mu_k$. We therefore conclude the proof of Lemma 2.11. □

3. Compactness

Let V be a K–R potential. In what follows, we set $f(x,s) = as^- + bs^+ + g(x,s)$, $g(x,0) = 0$, $\lim\limits_{|s|\to\infty} \frac{g(x,s)}{s} = 0$ uniformly with respect to $x \in \mathbb{R}^N$, and also make the following hypotheses:

(f_1) $\exists \beta > 0$, such that $\forall s_1, s_2 \in \mathbb{R}$, $\forall x \in \mathbb{R}^N$,

$$|g(x,s_1) - g(x,s_2)| \leq \beta |s_1 - s_2|$$

and $\beta < \sigma_0 - \Lambda(a,b)$, where $\Lambda(a,b) = \max\{a,b\}$.

It might just as well assume $m + \mu_1 > 0$ and $\inf \sigma\left(\widehat{A}_m\right) > 1$ with $\widehat{A}_m = -\Delta + 2(V+m)$ by choosing $m > 0$ suitably large. Clearly, $A_m := -\Delta + V + m$ is a positive definite self-adjoint operator with $D(A_m) = H^2\left(\mathbb{R}^N\right)$.

Denote by

$$H_m^1\left(\mathbb{R}^N\right) = \left\{u \in H^1\left(\mathbb{R}^N\right) : \int\limits_{\mathbb{R}^N} |\nabla u|^2 + (V+m)u^2 < +\infty\right\}$$

the Hilbert space equipped with the norm

$$\|u\|_m = \left[\int\limits_{\mathbb{R}^N} |\nabla u|^2 + (V+m)u^2\right]^{\frac{1}{2}}$$

and the inner product

$$\langle u, v\rangle_m = \int\limits_{\mathbb{R}^N} \nabla u \nabla v + (V+m)\, uv.$$

It is easy to check that $H_m^1\left(\mathbb{R}^N\right) = H^1\left(\mathbb{R}^N\right)$. Indeed, notice that

$$\begin{aligned}\frac{\|u\|^2}{2} &\le \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 + \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla u|^2 + 2\,(V+m)\,u^2 \\ &= \|u\|_m^2 \le \left(C\,\|V_1\|_{L^p} + \|V_2\|_{L^\infty} + m + 1\right)\|u\|^2,\end{aligned} \tag{3.1}$$

where $p=2$ if $N \le 3$, and $p > \frac{N}{2}$ if $N \ge 4$, so the assertion follows.

On account of the hypothesis, $\sigma\,(A_m) \in (0, +\infty)$, alluding to the fact that A_m^{-1} exists. Set $g_m\,(x,u) = f\,(x,u) + mu$. Obviously, in view of (f_1), $A_m^{-1} g_m\,(x,u) \in H^2$ for $\forall u \in H^1$. Hence, for fixed $\varphi \in H^1$,

$$\begin{aligned}\int\limits_{\mathbb{R}^N} g_m\,(x,u)\,\varphi &= \int\limits_{\mathbb{R}^N} A_m A_m^{-1} g_m\,(x,u)\cdot\varphi \\ &= \int\limits_{\mathbb{R}^N} \nabla A_m^{-1} g_m\,(x,u)\cdot\nabla\varphi + (V+m)\,A_m^{-1} g_m\,(x,u)\cdot\varphi.\end{aligned} \tag{3.2}$$

Therefore,

$$\begin{aligned}\left\langle J'(u), \varphi\right\rangle_m &= \left\langle u - A_m^{-1} g_m\,(x,u), \varphi\right\rangle_m \\ &= \int\limits_{\mathbb{R}^N} \nabla u \nabla\varphi + V\,(x)\,u\varphi - \int\limits_{\mathbb{R}^N} f\,(x,u)\,\varphi,\end{aligned} \tag{3.3}$$

yielding $J'(u) = u - A_m^{-1} g_m\,(x,u)$.

To show the main consequence of this section, we first recall the concept of Palais–Smale condition:

Definition 3.1. Let X be a Hilbert space, $\varphi \in C^1(X,\mathbb{R})$ and $c \in \mathbb{R}$. We say φ satisfies the $(\mathrm{PS})_c$ condition provided that any sequence $\{u_n\}_{n=1}^{+\infty} \subset X$ such that $\varphi(u_n) \to c$ and $\varphi'(u_n) \to 0$ in X, has a convergent subsequence in X.

Definition 3.2. Let X be a Hilbert space, $\varphi \in C^1(X,\mathbb{R})$. For $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, we say φ satisfies the (PS) condition on $[c_1,c_2]$ if φ satisfies the $(\mathrm{PS})_c$ condition for $\forall c \in [c_1,c_2]$.

Definition 3.3. Let X be a Hilbert space, $\varphi \in C^1(X,\mathbb{R})$. We say φ satisfies the (PS) condition provided that any sequence $\{u_n\}_{n=1}^{+\infty} \subset X$ such that $\{\varphi(u_n)\}_{n=1}^{+\infty}$ is bounded and $\varphi'(u_n) \to 0$ in X, has a convergent subsequence in X.

Theorem 3.4. *Let V be a real K–R potential, $\sigma_{\mathrm{dis}}\,(A) \ne \emptyset$, $\inf\sigma\,(A) = \inf\sigma_{\mathrm{dis}}\,(A)$. Under the hypothesis (f_1), if $(a,b) \notin \Sigma\,(A)$, then J satisfies the (PS) condition on $H_m^1\left(\mathbb{R}^N\right)$.*

Before entering the proof of Theorem 3.4, we state a well-known result (see [41]) as follows:

Proposition 3.5. *Assume that* $|\Omega| < \infty$, $1 \le p, r < \infty$, $f \in C\left(\overline{\Omega} \times \mathbb{R}, \mathbb{R}\right)$ *and*

$$|f(x,u)| \le C\left(1 + |u|^{p/r}\right). \tag{3.4}$$

Then, for every $u \in L^p(\Omega)$, $f(\cdot,u) \in L^r(\Omega)$ *and the operator*

$$T : L^p(\Omega) \to L^r(\Omega) : u \longmapsto f(x,u)$$

is continuous.

Proof of Theorem 3.4. In view of the hypothesis (f_1), it would not hurt to assume $\Lambda(a,b) + \beta > \mu_1$, s.t., $\sigma_{\text{dis}}(A) \cap \left[\mu_1, \Lambda(a,b) + \beta\right] = \{\lambda_i\}_{i=1}^{l}$, $\lambda_1 = \mu_1 = \inf \sigma(A) < \lambda_2 < \cdots < \lambda_l$. Divide the proof into three steps:

Step one. We prove that for any bounded sequence $\{u_k\}_{k=1}^{+\infty} \subset H^1\left(\mathbb{R}^N\right)$, s.t., $u_k \rightharpoonup u_0$ in $H^1\left(\mathbb{R}^N\right)$, $P_{N_l} u_k \to u^*$ in $H^1\left(\mathbb{R}^N\right)$, $u_k \rightharpoonup \widetilde{u}$ in $L^2\left(\mathbb{R}^N\right)$, then $\widetilde{u} = u_0$ a.e. on $\mathbb{R}^N$, $u^* = P_{N_l} u_0$ a.e. on $\mathbb{R}^N$, where P_{N_l} is the orthogonal projection onto N_l in $L^2\left(\mathbb{R}^N\right)$.

Observe that $u_k \to u_0$ in $L^2_{\text{loc}}\left(\mathbb{R}^N\right)$, then for any bounded open domain $\Omega \subset \mathbb{R}^N$, $u_k \to u_0$ in $L^2(\Omega)$, and hence

$$\int_\Omega u_k \varphi \to \int_\Omega u_0 \varphi, \forall \varphi \in L^2(\Omega). \tag{3.5}$$

Therefore,

$$\int_{\mathbb{R}^N} u_k \varphi \to \int_{\mathbb{R}^N} u_0 \varphi, \forall \varphi \in C_0^\infty\left(\mathbb{R}^N\right). \tag{3.6}$$

As $u_k \rightharpoonup \widetilde{u}$ in $L^2\left(\mathbb{R}^N\right)$, we have

$$\int_{\mathbb{R}^N} u_k \varphi \to \int_{\mathbb{R}^N} \widetilde{u} \varphi, \forall \varphi \in L^2\left(\mathbb{R}^N\right), \tag{3.7}$$

and thus,

$$\int_{\mathbb{R}^N} u_k \varphi \to \int_{\mathbb{R}^N} \widetilde{u} \varphi, \forall \varphi \in C_0^\infty\left(\mathbb{R}^N\right). \tag{3.8}$$

(3.6) together with (3.8) shows

$$\int_{\mathbb{R}^N} \left(u_0 - \widetilde{u}\right) \varphi = 0, \forall \varphi \in C_0^\infty\left(\mathbb{R}^N\right). \tag{3.9}$$

Due to the fact that $C_0^\infty\left(\mathbb{R}^N\right)$ is dense in $L^2\left(\mathbb{R}^N\right)$, we obtain

$$\int\limits_{\mathbb{R}^N}(u_0-\widetilde{u})\,\varphi=0, \forall\varphi\in L^2\left(\mathbb{R}^N\right), \tag{3.10}$$

so $\widetilde{u}=u_0$ a.e. on $\mathbb{R}^N$. This indicates that

$$u_k \rightharpoonup u_0 \text{ in } L^2\left(\mathbb{R}^N\right), \tag{3.11}$$

and thereby,

$$\int\limits_{\mathbb{R}^N}P_{N_l}u_k\varphi\to\int\limits_{\mathbb{R}^N}P_{N_l}u_0\varphi, \forall\varphi\in L^2\left(\mathbb{R}^N\right). \tag{3.12}$$

Since $P_{N_l}u_k\to u^*$ in $H^1\left(\mathbb{R}^N\right)$ yields

$$\int\limits_{\mathbb{R}^N}P_{N_l}u_k\varphi\to\int\limits_{\mathbb{R}^N}u^*\varphi, \forall\varphi\in L^2\left(\mathbb{R}^N\right), \tag{3.13}$$

combining (3.12) with (3.13) we get $u^*=P_{N_l}u_0$ a.e. on $\mathbb{R}^N$. This concludes the proof.

Step two. We show that for any sequence $\{u_k\}_{k=1}^{+\infty}\subset H_m^1\left(\mathbb{R}^N\right)$, s.t. $\left\|J'(u_k)\right\|_m\to 0$, is bounded in $H_m^1\left(\mathbb{R}^N\right)$.

Suppose by contradiction, there exists $\{u_k\}_{k=1}^{+\infty}\subset H_m^1\left(\mathbb{R}^N\right)$, $\left\|J'(u_k)\right\|_m\to 0$, $\|u_k\|_m\to+\infty$, and so $\|u_k\|\to+\infty$. Take $M_k^*\to+\infty$, s.t. $\frac{M_k^*}{\|u_k\|}\to 0$. Set $\widetilde{u}_k=\frac{u_k}{\|u_k\|}$. We claim that for $\forall\varphi\in C_0^\infty\left(\mathbb{R}^N\right)$,

$$\lim_{k\to+\infty}\frac{\left|\int_{\mathbb{R}^N}g(x,u_k)\,\varphi\right|}{\|u_k\|}=0. \tag{3.14}$$

Actually, notice that $C_0^\infty\left(\mathbb{R}^N\right)\subset L^1\left(\mathbb{R}^N\right)$, so for $\forall\varphi\in C_0^\infty\left(\mathbb{R}^N\right)$, $\forall\varepsilon>0$, $\exists K_1^*=K_1^*(\varepsilon,\varphi)\in\mathbb{N}$, $\forall k\geq K_1^*$,

$$\begin{aligned}\int\limits_{|u_k|<M_k^*}\frac{|g(x,u_k)\,\varphi|}{\|u_k\|}&\leq\frac{\beta M_k^*}{\|u_k\|}\cdot\|\varphi\|_{L^1(\mathbb{R}^N)}\\&\leq\frac{\varepsilon}{2\,\|\varphi\|_{L^1(\mathbb{R}^N)}}\cdot\|\varphi\|_{L^1(\mathbb{R}^N)}=\frac{\varepsilon}{2}.\end{aligned} \tag{3.15}$$

On the other hand, $\forall\varepsilon>0$, $\exists S^*=S^*(\varepsilon,\varphi)>0$, $\forall s\in\mathbb{R}$, $|s|\geq S^*$,

$$\max_{x\in\mathbb{R}^N}\left|\frac{f(x,s)-as^- -bs^+}{s}\right|<\frac{\varepsilon}{2\,\|\varphi\|}, \tag{3.16}$$

and hence $\exists K_2^* = K_2^*(\varepsilon, \varphi) \in \mathbb{N}$, $\forall k \geq K_2^*$, $M_k^* > S^*$,

$$\begin{aligned} &\int\limits_{|u_k| \geq M_k^*} \frac{|g(x, u_k)\varphi|}{\|u_k\|} \\ &= \int\limits_{|u_k| \geq M_k^*} \left| \frac{g(x, u_k)}{u_k} \right| \cdot |\widetilde{u}_k| \cdot |\varphi| \\ &\leq \frac{\varepsilon}{2\|\varphi\|} \int\limits_{\mathbb{R}^N} |\widetilde{u}_k| \cdot |\varphi| \\ &\leq \frac{\varepsilon}{2\|\varphi\|} \left(\int\limits_{\mathbb{R}^N} |\widetilde{u}_k|^2 \right)^{\frac{1}{2}} \left(\int\limits_{\mathbb{R}^N} |\varphi|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{\varepsilon}{2\|\varphi\|} \cdot \|\varphi\| = \frac{\varepsilon}{2}. \end{aligned} \tag{3.17}$$

Therefore, for $\forall k \geq K^* = \Lambda\left(K_1^*, K_2^*\right)$,

$$\begin{aligned} &\frac{\left| \int_{\mathbb{R}^N} g(x, u_k)\varphi \right|}{\|u_k\|} \\ &\leq \int\limits_{|u_k| < M_k^*} \frac{|g(x, u_k)\varphi|}{\|u_k\|} + \int\limits_{|u_k| \geq M_k^*} \frac{|g(x, u_k)\varphi|}{\|u_k\|} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned} \tag{3.18}$$

and this alludes to the claim.

Assume $\widetilde{u}_k \rightharpoonup u_0$ in $H^1\left(\mathbb{R}^N\right)$, then $\widetilde{u}_k \to u_0$ in $L^p_{\mathrm{loc}}\left(\mathbb{R}^N\right)$, $2 \leq p < 2^*$. According to the definition, for fixed $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$, there is a bounded open domain $\Omega \subset \mathbb{R}^N$ such that $\left\{x \in \mathbb{R}^N : \varphi(x) \neq 0\right\} \subset \Omega$. Therefore, $\widetilde{u}_k \to u_0$ in $L^2(\Omega)$. Since $\frac{\|J'(u_k)\|_m}{\|u_k\|} \to 0$, by (3.14), we have

$$\int\limits_{\mathbb{R}^N} \nabla u_0 \nabla \varphi + V(x) u_0 \varphi = a \int\limits_{\mathbb{R}^N} u_0^- \varphi + b \int\limits_{\mathbb{R}^N} u_0^+ \varphi, \forall \varphi \in C_0^\infty\left(\mathbb{R}^N\right). \tag{3.19}$$

Now we show that

$$\int\limits_{\mathbb{R}^N} \nabla u_0 \nabla \varphi + V(x) u_0 \varphi = a \int\limits_{\mathbb{R}^N} u_0^- \varphi + b \int\limits_{\mathbb{R}^N} u_0^+ \varphi, \forall \varphi \in H^1\left(\mathbb{R}^N\right), \tag{3.20}$$

and this indeed indicates that u_0 is a weak solution of (1.3).

Notice that $C_0^\infty\left(\mathbb{R}^N\right)$ is dense in $H^1\left(\mathbb{R}^N\right)$, so for fixed $\varphi\in H^1\left(\mathbb{R}^N\right)$, we assume $\{\varphi_k\}_{k=1}^{+\infty}\subset C_0^\infty\left(\mathbb{R}^N\right)$, $\varphi_k\to\varphi$ in $H^1\left(\mathbb{R}^N\right)$. An argument analogous to the combination of (2.16) and (2.17) gets

$$\lim_{k\to+\infty}\int\limits_{\mathbb{R}^N} V(x)\,u_0\varphi_k=\int\limits_{\mathbb{R}^N} V(x)\,u_0\varphi, \tag{3.21}$$

alluding to (3.20).

Since $(a,b)\notin\Sigma(A)$, we derive $u_0=0$. Now we claim that $\|\widetilde{u}_k\|_{L^2}\to 0$. Otherwise, $\exists\delta^*>0$, $\exists K_1\in\mathbb{N}$, $\forall k\in\mathbb{N}$, $k\geq K_1$, $\|\widetilde{u}_k\|_{L^2}\geq\delta^*$. In view of the hypotheses, it is easy to verify the following fact

$$\sup_{s_1,\,s_2\in\mathbb{R},\,s_1\neq s_2}\ \sup_{x\in\mathbb{R}^N}\frac{f(x,s_1)-f(x,s_2)}{s_1-s_2}\leq\Lambda(a,b)+\beta. \tag{3.22}$$

Notice that $\frac{|\langle J'(u_k),u_k\rangle_m|}{\|u_k\|^2}\to 0$, hence, for $\forall\varepsilon>0$, $\exists K_2\in\mathbb{N}$, $\forall k\in\mathbb{N}$, $k\geq K_2$,

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}\left|\nabla P_{N_l}\widetilde{u}_k\right|^2+V(x)\left(P_{N_l}\widetilde{u}_k\right)^2\\&+\int\limits_{\mathbb{R}^N}\left|\nabla P_{N_l^\perp}\widetilde{u}_k\right|^2+V(x)\left(P_{N_l^\perp}\widetilde{u}_k\right)^2\\&\leq\int\limits_{u_k\neq 0}\frac{f(x,u_k)}{u_k}\widetilde{u}_k^2+\varepsilon\leq(\Lambda(a,b)+\beta)\|\widetilde{u}_k\|_{L^2}^2+\varepsilon,\end{aligned} \tag{3.23}$$

where $P_{N_l^\perp}$ is the orthogonal projection onto $N_l^\perp$ in $L^2\left(\mathbb{R}^N\right)$.

The hypothesis shows $\Lambda(a,b)+\beta<\theta_l:=\Gamma\left(\mu_{d_l+1},\sigma_0\right)=\min\left\{\mu_{d_l+1},\sigma_0\right\}$, and then there exists $d^*\in(\Lambda(a,b)+\beta,\theta_l)$, for $\varepsilon>0$ sufficiently small and $\forall k\in\mathbb{N}$, $k\geq K=\Lambda(K_1,K_2)$,

$$(\Lambda(a,b)+\beta)\|\widetilde{u}_k\|_{L^2}^2+\varepsilon\leq d^*\|\widetilde{u}_k\|_{L^2}^2. \tag{3.24}$$

Inserting (3.24) into (3.23),

$$\begin{aligned}&\int\limits_{\mathbb{R}^N}\left|\nabla P_{N_l}\widetilde{u}_k\right|^2+V(x)\left(P_{N_l}\widetilde{u}_k\right)^2\\&+\int\limits_{\mathbb{R}^N}\left|\nabla P_{N_l^\perp}\widetilde{u}_k\right|^2+V(x)\left(P_{N_l^\perp}\widetilde{u}_k\right)^2\\&\leq d^*\left\|P_{N_l}\widetilde{u}_k\right\|_{L^2}^2+d^*\left\|P_{N_l^\perp}\widetilde{u}_k\right\|_{L^2}^2,\end{aligned} \tag{3.25}$$

and therefore by Lemma 2.11,

$$\begin{aligned}
&\left(\theta_l - d^*\right) \left\| P_{N_l^\perp} \widetilde{u}_k \right\|_{L^2}^2 \\
&\leq \int_{\mathbb{R}^N} \left| \nabla P_{N_l^\perp} \widetilde{u}_k \right|^2 + V(x) \left(P_{N_l^\perp} \widetilde{u}_k \right)^2 - d^* \left\| P_{N_l^\perp} \widetilde{u}_k \right\|_{L^2}^2 \\
&\leq d^* \left\| P_{N_l} \widetilde{u}_k \right\|_{L^2}^2 - \int_{\mathbb{R}^N} \left| \nabla P_{N_l} \widetilde{u}_k \right|^2 + V(x) \left(P_{N_l} \widetilde{u}_k \right)^2 \\
&\leq \left(d^* - \mu_1 \right) \left\| P_{N_l} \widetilde{u}_k \right\|_{L^2}^2 ,
\end{aligned} \tag{3.26}$$

i.e.,

$$\left\| P_{N_l^\perp} \widetilde{u}_k \right\|_{L^2}^2 \leq \frac{d^* - \mu_1}{\theta_l - d^*} \left\| P_{N_l} \widetilde{u}_k \right\|_{L^2}^2 . \tag{3.27}$$

As $u_0 = 0$, by step one, $P_{N_l} \widetilde{u}_k \to 0$ in $H^1\left(\mathbb{R}^N\right)$, and thus

$$P_{N_l} \widetilde{u}_k \to 0 \text{ in } L^2\left(\mathbb{R}^N\right). \tag{3.28}$$

Thereby, it follows from (3.27) that

$$P_{N_l^\perp} \widetilde{u}_k \to 0 \text{ in } L^2\left(\mathbb{R}^N\right). \tag{3.29}$$

(3.28) + (3.29) $\Rightarrow \widetilde{u}_k \to 0$ in $L^2\left(\mathbb{R}^N\right)$, which contradicts original hypothesis $\|\widetilde{u}_k\|_{L^2} \geq \delta^*$ for $k \geq K_1$. We arrive at the claim as desired. By [23], for $\forall r \in (2, 2^*)$,

$$\|\widetilde{u}_k\|_{L^r} \to 0, \tag{3.30}$$

and accordingly,

$$\begin{aligned}
1 &= \frac{\left\langle J'(u_k), u_k \right\rangle_m}{\|u_k\|^2} - \int_{\mathbb{R}^N} V(x) \widetilde{u}_k^2 \\
&\quad + \|\widetilde{u}_k\|_{L^2}^2 + \int_{\{x \in \mathbb{R}^N :\, u_k \neq 0\}} \frac{f(x, u_k)}{u_k} \widetilde{u}_k^2 \\
&\leq \left\| J'(u_k) \right\|_m \cdot \frac{\|u_k\|_m}{\|u_k\|^2} + \|V_1\|_{L^p} \cdot \|\widetilde{u}_k\|_{L^{2q}}^2 \\
&\quad + \left(\|V_2\|_{L^\infty} + |\Lambda(a, b)| + \beta + 1 \right) \cdot \|\widetilde{u}_k\|_{L^2}^2 \\
&\to 0
\end{aligned} \tag{3.31}$$

as $k \to +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > \frac{N}{2}$ if $N \geq 4$, and $p = 2$ if $N \leq 3$. We get a contradiction! The proof is complete.

Step three. We show that any sequence $\{u_k\}_{k=1}^{+\infty} \subset H_m^1(\mathbb{R}^N)$, s.t. $\|J'(u_k)\|_m \to 0$, contains a convergent subsequence. Since $\{u_k\}_{k=1}^{+\infty}$ is bounded in $H^1(\mathbb{R}^N)$ by step two, we assume $u_k \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$. For $\varphi \in C_0^\infty(\mathbb{R}^N)$, there is a bounded domain $\Omega \subset \mathbb{R}^N$, $\{x \in \mathbb{R}^N : \varphi(x) \neq 0\} \subset \Omega$. Hence, $u_k \to u_0$ in $L^2(\Omega)$. As

$$|f(x,s)| \le (\Lambda(|a|,|b|)+\beta)\cdot(1+|s|), \tag{3.32}$$

for $\forall x \in \mathbb{R}^N$, $\forall s \in \mathbb{R}$, by Proposition 3.5,

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} \left[f(x,u_k) - f(x,u_0)\right]\cdot\varphi\right| \\ &\le \|f(x,u_k) - f(x,u_0)\|_{L^2(\Omega)} \cdot \|\varphi\|_{L^2(\Omega)} \to 0, \end{aligned} \tag{3.33}$$

and a standard argument shows

$$\lim_{k\to+\infty} \int_{\mathbb{R}^N} V(x)\,u_k\varphi = \int_{\mathbb{R}^N} V(x)\,u_0\varphi. \tag{3.34}$$

Consequently,

$$\int_{\mathbb{R}^N} \nabla u_0 \nabla\varphi + V(x)\,u_0\varphi = \int_{\mathbb{R}^N} f(x,u_0)\,\varphi, \forall\varphi \in C_0^\infty\left(\mathbb{R}^N\right). \tag{3.35}$$

Based on the density of $C_0^\infty(\mathbb{R}^N)$ in $H^1(\mathbb{R}^N)$, we yield

$$\int_{\mathbb{R}^N} \nabla u_0 \nabla\varphi + V(x)\,u_0\varphi = \int_{\mathbb{R}^N} f(x,u_0)\,\varphi, \forall\varphi \in H^1\left(\mathbb{R}^N\right), \tag{3.36}$$

and this indicates that u_0 is a weak solution of (1.1).

Now we remain to prove $u_k \to u_0$ in $H^1(\mathbb{R}^N)$. Note that

$$\begin{aligned} &\left\langle J'(u_k) - J'(u_0), u_k - u_0\right\rangle_m \\ &= \int_{\mathbb{R}^N} |\nabla(u_k-u_0)|^2 + V(x)(u_k-u_0)^2 \\ &\qquad - \int_{\mathbb{R}^N} \left[f(x,u_k) - f(x,u_0)\right]\cdot(u_k-u_0) \\ &\to 0, \end{aligned} \tag{3.37}$$

hence, by (3.37), for $\forall\varepsilon > 0$, $\exists K^* \in \mathbb{N}$, $\forall k \in \mathbb{N}$, $k \ge K^*$, such that

$$
\begin{aligned}
&\int_{\mathbb{R}^N} \left|\nabla P_{N_l}(u_k-u_0)\right|^2 + V(x)\left|P_{N_l}(u_k-u_0)\right|^2 \\
&\quad + \int_{\mathbb{R}^N} \left|\nabla P_{N_l^\perp}(u_k-u_0)\right|^2 + V(x)\left|P_{N_l^\perp}(u_k-u_0)\right|^2 \\
&\le \int_{\mathbb{R}^N} \left[f(x,u_k)-f(x,u_0)\right]\cdot(u_k-u_0)+\varepsilon \\
&\le (\Lambda(a,b)+\beta)\|u_k-u_0\|_{L^2}^2+\varepsilon. \qquad (3.38)
\end{aligned}
$$

Mimicking the trick employed by preceding argument, we derive

$$u_k \to u_0 \text{ in } L^2\left(\mathbb{R}^N\right). \qquad (3.39)$$

Once again by [23], for $\forall r\in(2,2^*)$,

$$u_k \to u_0 \text{ in } L^r\left(\mathbb{R}^N\right). \qquad (3.40)$$

Using (3.37), (3.39) and (3.40) we have

$$
\begin{aligned}
\|u_k-u_0\|^2 \le\ & \left\|J'(u_k)\right\|_m\cdot\|u_k-u_0\|_m+\|V_1\|_{L^p}\cdot\|u_k-u_0\|_{L^{2q}}^2 \\
&+(\|V_2\|_{L^\infty}+|\Lambda(a,b)|+\beta+1)\cdot\|u_k-u_0\|_{L^2}^2 \\
\to\ & 0, \qquad (3.41)
\end{aligned}
$$

$\frac{1}{p}+\frac{1}{q}=1$, $p>\frac{N}{2}$ if $N\ge 4$, and $p=2$ if $N\le 3$. The assertion follows. □

4. Convexity and concavity

Before presenting section, we first recall some definitions:

Definition 4.1. Let E be a Hilbert space and $\mu>0$ be a number. A map $h:E\to E$ is said to be μ-monotone on E if and only if

$$\langle h(u)-h(v),u-v\rangle_E \ge \mu\|u-v\|_E^2,\ \forall u,v\in E. \qquad (4.1)$$

Definition 4.2. Let E be a Hilbert space. A function $I:E\to\mathbb{R}$ is said to be μ-convex on E if and only if for $\forall t\in[0,1]$, $\forall u,v\in E$,

$$I((1-t)u+tv)+\frac{\mu}{2}t(1-t)\|u-v\|_E^2 \le (1-t)I(u)+tI(v). \qquad (4.2)$$

Assume:

(V^*): Let V be a K–R potential, $\sigma_{\text{dis}}(A)\neq\emptyset$, $\inf\sigma(A)=\inf\sigma_{\text{dis}}(A)$, $\{\lambda_i\}_{i=1}^l\in\sigma_{\text{dis}}(A)$, $\sigma_{\text{dis}}(A)\cap[\mu_1,\lambda_l]=\{\lambda_i\}_{i=1}^l$, $l\ge 1$, s.t., $\lambda_1=\mu_1=\inf\sigma(A)<\lambda_2<\cdots<\lambda_l<\sigma_0$.

Lemma 4.3. *Under the hypothesis* (V^*)*, then for fixed* $v \in N_l$*,* $I(v+w,a,b)$ *is* μ*-convex in* $w \in M_l$ *if and only if* $w_1, w_2 \in M_l$*,*

$$\left\langle I'(v+w_1,a,b) - I'(v+w_2,a,b), w_1 - w_2 \right\rangle_m \geq \mu \|w_1 - w_2\|_m^2 . \tag{4.3}$$

Proof. "Only if". According to the definition, if $I(v+w,a,b)$ is μ-convex in $w \in M_l$, then for fixed $v \in N_l$, $\forall t \in [0,1]$, $\forall w_1, w_2 \in M_l$,

$$\begin{aligned} &I(v+(1-t)w_1 + tw_2, a, b) + \frac{\mu}{2} t(1-t)\|w_1 - w_2\|_m^2 \\ &\leq (1-t) I(v+w_1,a,b) + tI(v+w_2,a,b), \end{aligned} \tag{4.4}$$

i.e.,

$$\begin{aligned} &I(v+w_1+t(w_2-w_1),a,b) - I(v+w_1,a,b) + \frac{\mu}{2} t(1-t)\|w_1 - w_2\|_m^2 \\ &\leq t\left[I(v+w_2,a,b) - I(v+w_1,a,b)\right]. \end{aligned} \tag{4.5}$$

Therefore,

$$\begin{aligned} &\int_0^1 \left\langle I'(v+w_1+st(w_2-w_1),a,b), w_2 - w_1 \right\rangle_m ds + \frac{\mu}{2}(1-t)\|w_1 - w_2\|_m^2 \\ &\leq I(v+w_2,a,b) - I(v+w_1,a,b). \end{aligned} \tag{4.6}$$

Make use of the integral theorem of the mean, $\exists s_1 \in (0,1)$,

$$\begin{aligned} &\left\langle I'(v+w_1+s_1t(w_2-w_1),a,b), w_2 - w_1 \right\rangle_m + \frac{\mu}{2}(1-t)\|w_1 - w_2\|_m^2 \\ &\leq I(v+w_2,a,b) - I(v+w_1,a,b). \end{aligned} \tag{4.7}$$

Proceeding along the same lines,

$$\begin{aligned} &\int_0^1 \left\langle I'(v+w_2+st(w_1-w_2),a,b), w_1 - w_2 \right\rangle_m ds + \frac{\mu}{2}(1-t)\|w_1 - w_2\|_m^2 \\ &\leq I(v+w_1,a,b) - I(v+w_2,a,b), \end{aligned} \tag{4.8}$$

and thus, $\exists s_2 \in (0,1)$,

$$\begin{aligned} &\left\langle I'(v+w_2+s_2t(w_1-w_2),a,b), w_1 - w_2 \right\rangle_m + \frac{\mu}{2}(1-t)\|w_1 - w_2\|_m^2 \\ &\leq I(v+w_1,a,b) - I(v+w_2,a,b). \end{aligned} \tag{4.9}$$

Let $t \to 0$, hence, the combination of (4.7) and (4.9) obtains

$$\left.\begin{aligned}\left\langle I'(v+w_1,a,b),w_2-w_1\right\rangle_m+\tfrac{\mu}{2}\|w_1-w_2\|_m^2\le I(v+w_2,a,b)-I(v+w_1,a,b),\\ \left\langle I'(v+w_2,a,b),w_1-w_2\right\rangle_m+\tfrac{\mu}{2}\|w_1-w_2\|_m^2\le I(v+w_1,a,b)-I(v+w_2,a,b),\end{aligned}\right\} \tag{4.10}$$

and this gets (4.3).

"If". Observe that (4.4) is equivalent to (4.6), and (4.6) can be replaced by

$$\begin{aligned}&\int_0^1 \left\langle I'(v+w_1+st(w_2-w_1),a,b),w_2-w_1\right\rangle_m ds+\frac{\mu}{2}(1-t)\|w_1-w_2\|_m^2\\ &\le \int_0^1 \left\langle I'(v+w_1+s(w_2-w_1),a,b),w_2-w_1\right\rangle_m ds,\end{aligned} \tag{4.11}$$

yielding

$$\begin{aligned}&\int_0^1 \left\langle I'(v+w_1+s(w_2-w_1),a,b)-I'(v+w_1+st(w_2-w_1),a,b),w_2-w_1\right\rangle_m ds\\ &\ge \frac{\mu}{2}(1-t)\|w_1-w_2\|_m^2.\end{aligned} \tag{4.12}$$

(4.12) is clearly derived by (4.3). We conclude the proof. □

Lemma 4.4. *Under the hypothesis* (V^*), *if* $\Lambda(a,b)<\theta_l=\Gamma\left(\mu_{d_l+1},\sigma_0\right)$, *then for fixed* $v\in N_l$, $I'_w(v+w,a,b)$ *is* μ*-monotone on* M_l, *i.e.,* $\exists\mu>0$, *for* $\forall w_1,w_2\in M_l$,

$$\left\langle I'_w(v+w_1,a,b)-I'_w(v+w_2,a,b),w_1-w_2\right\rangle_m\ge\mu\|w_1-w_2\|_m^2. \tag{4.13}$$

Proof. Due to the facts

$$(v+w_1)^+-(v+w_2)^+\le(w_1-w_2)^+, \tag{4.14}$$

$$(v+w_1)^+-(v+w_2)^+\ge(w_1-w_2)^-, \tag{4.15}$$

$$(v+w_1)^--(v+w_2)^-\ge(w_1-w_2)^-, \tag{4.16}$$

$$(v+w_1)^--(v+w_2)^-\le(w_1-w_2)^+, \tag{4.17}$$

we have

$$\begin{aligned}&\left\langle I'(v+w_1,a,b)-I'(v+w_2,a,b),w_1-w_2\right\rangle_m\\ &=\int_{\mathbb{R}^N}|\nabla(w_1-w_2)|^2+V(x)(w_1-w_2)^2-a\|w_1-w_2\|_{L^2}^2\\ &\quad-(b-a)\int_{\mathbb{R}^N}\left[(v+w_1)^+-(v+w_2)^+\right]\cdot(w_1-w_2)^+\end{aligned}$$

$$
\begin{aligned}
&- (b-a) \int_{\mathbb{R}^N} \left[(v+w_1)^+ - (v+w_2)^+ \right] \cdot (w_1-w_2)^- \\
\geq &\int_{\mathbb{R}^N} |\nabla (w_1-w_2)|^2 + V(x)(w_1-w_2)^2 - a \|w_1-w_2\|_{L^2}^2 \\
&- (b-a) \left\| (w_1-w_2)^+ \right\|_{L^2}^2 - (b-a) \left\| (w_1-w_2)^- \right\|_{L^2}^2 \\
\geq &\frac{\theta_l - b}{\theta_l + m} \|w_1-w_2\|_m^2
\end{aligned}
\tag{4.18}
$$

if $a \leq b$, and

$$
\begin{aligned}
&\left\langle I'(v+w_1,a,b) - I'(v+w_2,a,b), w_1-w_2 \right\rangle_m \\
= &\int_{\mathbb{R}^N} |\nabla (w_1-w_2)|^2 + V(x)(w_1-w_2)^2 - b \|w_1-w_2\|_{L^2}^2 \\
&- (a-b) \int_{\mathbb{R}^N} \left[(v+w_1)^- - (v+w_2)^- \right] \cdot (w_1-w_2)^+ \\
&- (a-b) \int_{\mathbb{R}^N} \left[(v+w_1)^- - (v+w_2)^- \right] \cdot (w_1-w_2)^- \\
\geq &\int_{\mathbb{R}^N} |\nabla (w_1-w_2)|^2 + V(x)(w_1-w_2)^2 - b \|w_1-w_2\|_{L^2}^2 \\
&- (a-b) \left\| (w_1-w_2)^+ \right\|_{L^2}^2 - (a-b) \left\| (w_1-w_2)^- \right\|_{L^2}^2 \\
\geq &\frac{\theta_l - a}{\theta_l + m} \|w_1-w_2\|_m^2
\end{aligned}
\tag{4.19}
$$

if $b \leq a$. This ends (4.13) as desired. □

Using Lemma 4.3 and Lemma 4.4 we have

Corollary 4.5. *Under the hypotheses of Lemma 4.4, for fixed $v \in N_l$, $I(v+w,a,b)$ is μ-convex in $w \in M_l$.*

Definition 4.6. Let E be a Hilbert space and $\rho > 0$ be a number. A map $h : E \to E$ is said to be ρ-antimonotone on E if and only if

$$
\langle h(u) - h(v), u - v \rangle_E \leq -\rho \|u-v\|_E^2, \forall u, v \in E. \tag{4.20}
$$

Definition 4.7. Let E be a Hilbert space. A function $I : E \to \mathbb{R}$ is said to be ρ-concave on E if and only if for $\forall t \in [0,1]$, $\forall u, v \in E$,

$$
I((1-t)u + tv) - \frac{\rho}{2} t(1-t) \|u-v\|_E^2 \geq (1-t) I(u) + t I(v). \tag{4.21}
$$

An argument quite similar to Lemma 4.3 yields

Lemma 4.8. *Under the hypothesis* (V^*), *for fixed* $w \in M_l$, $I(v+w,a,b)$ *is* ρ*-concave in* $v \in N_l$ *if and only if* $\forall v_1, v_2 \in N_l$,

$$\left\langle I'(v_1+w,a,b) - I'(v_2+w,a,b), v_1-v_2 \right\rangle_m \le -\rho \|v_1-v_2\|_m^2. \tag{4.22}$$

Lemma 4.9. *Under the hypothesis* (V^*), *if* $\Gamma(a,b) > \lambda_l$, *then for fixed* $w \in M_l$, $I'_v(v+w,a,b)$ *is* ρ*-antimonotone on* N_l, *i.e.,* $\exists \rho > 0$, *for* $\forall v_1, v_2 \in N_l$,

$$\left\langle I'_v(v_1+w,a,b) - I'_v(v_2+w,a,b), v_1-v_2 \right\rangle_m \le -\rho \|v_1-v_2\|_m^2. \tag{4.23}$$

Proof. An argument analogous to the proof of Lemma 4.4 indicates that

$$\begin{aligned}
&\left\langle I'(v_1+w,a,b) - I'(v_2+w,a,b), v_1-v_2 \right\rangle_m \\
&= \int_{\mathbb{R}^N} |\nabla(v_1-v_2)|^2 + V(x)(v_1-v_2)^2 - a\|v_1-v_2\|_{L^2}^2 \\
&\quad - (b-a)\int_{\mathbb{R}^N} \left((v_1+w)^+ - (v_2+w)^+\right)(v_1-v_2)^+ \\
&\quad - (b-a)\int_{\mathbb{R}^N} \left((v_1+w)^+ - (v_2+w)^+\right)(v_1-v_2)^- \\
&\le \int_{\mathbb{R}^N} |\nabla(v_1-v_2)|^2 + V(x)(v_1-v_2)^2 - a\|v_1-v_2\|_{L^2}^2 \\
&\le -\frac{a-\lambda_l}{\lambda_l+m}\|v_1-v_2\|_m^2,
\end{aligned} \tag{4.24}$$

as $a \le b$, and

$$\begin{aligned}
&\left\langle I'(v_1+w,a,b) - I'(v_2+w,a,b), v_1-v_2 \right\rangle_m \\
&= \int_{\mathbb{R}^N} |\nabla(v_1-v_2)|^2 + V(x)(v_1-v_2)^2 - b\|v_1-v_2\|_{L^2}^2 \\
&\quad - (a-b)\int_{\mathbb{R}^N} \left[(v_1+w)^- - (v_2+w)^-\right]\cdot(v_1-v_2)^+ \\
&\quad - (a-b)\int_{\mathbb{R}^N} \left[(v_1+w)^- - (v_2+w)^-\right]\cdot(v_1-v_2)^- \\
&\le \int_{\mathbb{R}^N} |\nabla(v_1-v_2)|^2 + V(x)(v_1-v_2)^2 - b\|v_1-v_2\|_{L^2}^2 \\
&\le -\frac{b-\lambda_l}{\lambda_l+m}\|v_1-v_2\|_m^2
\end{aligned} \tag{4.25}$$

as $b \le a$, fulfilling the prophecy (4.23). □

Resorting to Lemma 4.8 and Lemma 4.9 we get

Corollary 4.10. *Under the hypotheses of Lemma 4.9, for fixed $w \in M_l$, $I(v+w,a,b)$ is ρ-concave in $v \in N_l$, i.e., for $\forall t \in [0,1]$, $\forall v_1, v_2 \in N_l$,*

$$\begin{aligned} &I((1-t)v_1 + tv_2 + w, a, b) - \frac{\rho}{2} t(1-t)\|u-v\|_m^2 \\ &\geq (1-t) I(v_1+w,a,b) + tI(v_2+w,a,b). \end{aligned} \tag{4.26}$$

5. The Fučík spectrum of Schrödinger operator

5.1. *Computation of critical groups $C_*(I,0)$*

Review the definitions of the following quantities

$$F_{1l}(w,a,b) = \sup\{I(v+w,a,b) : v \in N_l\}, \tag{5.1}$$

$$F_{2l}(v,a,b) = \inf\{I(v+w,a,b) : w \in M_l\}, \tag{5.2}$$

$$R_l(a,b) = \inf\{F_{1l}(w,a,b) : w \in M_l\}, \tag{5.3}$$

$$r_l(a,b) = \sup\{F_{2l}(v,a,b) : v \in N_l\}, \tag{5.4}$$

$$M_l(a,b) = \inf\{F_{1l}(w,a,b) : w \in M_l, \|w\| = 1\}, \tag{5.5}$$

$$m_l(a,b) = \sup\{F_{2l}(v,a,b) : v \in N_l, \|v\| = 1\}, \tag{5.6}$$

$$\nu_l(a) = \sup\{b : M_l(a,b) \geq 0\}, \tag{5.7}$$

$$\mu_l(a) = \inf\{b : m_l(a,b) \leq 0\}, \tag{5.8}$$

where (5.1)–(5.4) were introduced by [35], and quantities (5.5)–(5.8) were introduced by Cac [7] (with some changes in notation).

Let E be a Hilbert space and $f \in C^1(E,\mathbb{R})$. Set $K_f = \{u \in E : f'(u) = 0\}$ for the set of critical points of f on E, and assume that u_0 is an isolated critical point of f on E, $c = f(u_0)$.

Recall that the q-th critical group, with coefficient group G of f at u_0, is defined by

$$C_q(f,u_0) = H_q\left(f^c \cap U, f^c \cap U \setminus \{u_0\}, G\right), \tag{5.9}$$

where $f^c = \{u \in E : f(u) \leq c\}$ denotes the sublevel sets as usual, and U is any neighborhood of u_0 such that u_0 is the only critical point of f in $f^c \cap U$, and $H_*(X,Y,G)$ represents the singular relative homology groups with the abelian coefficient group G. According to the definition, if $(a,b) \notin \Sigma(A)$, 0 is the only critical point of I in $H^1(\mathbb{R}^N)$, and hence we can take $U = H^1(\mathbb{R}^N)$. Then

$$C_q(I,0) = H_q\left(I^0, I^0 \setminus \{0\}\right). \tag{5.10}$$

For the convenience of later discussion, an additional assumption on $V(x)$ is presented here: (V_2) $V(x)$ is a real potential, s.t.,

$$\inf_{x\in\mathbb{R}^N} V(x) > -\infty. \tag{5.11}$$

Lemma 5.1. *Under the hypothesis* (V^*), *if*

(i) $\Lambda(a,b) < \theta_l$;

or

(ii) (V_2) *follows, and* $\Gamma(a,b) < \Lambda(a,b) = \theta_l < \sigma_0$,

then for each $v \in N_l$, $F_{2l}(v,a,b)$ *can be achieved and every minimizing sequence contains a convergent subsequence. Moreover, if* $\Lambda(a,b) < \theta_l$, *the minimizer is unique.*

Proof. First of all, we show that for fixed $v \in N_l$, $I(v+w,a,b)$ is coercive on M_l, i.e., for any sequence $\{w_i^*\}_{i=1}^{+\infty} \subset M_l$,

$$I(v+w_i^*,a,b) \to +\infty \tag{5.12}$$

as $\|w_i^*\|_m \to +\infty$. By way of negation, there exist $M>0$ and a renamed sequence $\{w_i^*\}_{i=1}^{+\infty} \subset M_l$, s.t.

$$I(v+w_i^*,a,b) \le M \tag{5.13}$$

as $\|w_i^*\|_m \to +\infty$.

If $\Lambda(a,b) < \theta_l$, we have

$$\begin{aligned} M &\ge I\left(v+w_i^*,a,b\right) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)v^2 + \frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla w_i^*\right|^2 + V(x)\left|w_i^*\right|^2 \\ &\quad -\frac{a}{2}\left\|\left(v+w_i^*\right)^-\right\|_{L^2}^2 - \frac{b}{2}\left\|\left(v+w_i^*\right)^+\right\|_{L^2}^2 \\ &\ge \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)v^2 - \frac{\Lambda(a,b)}{2}\|v\|_{L^2}^2 \\ &\quad + \frac{\theta_l - \Lambda(a,b)}{2(\theta_l+m)}\cdot\left\|w_i^*\right\|_m^2 \end{aligned} \tag{5.14}$$

and so we get the boundedness of $\|w_i^*\|_m$, contradicting $\|w_i^*\|_m \to +\infty$.

In the case of $\Lambda(\lambda_1, \Gamma(a,b)) < \Lambda(a,b) = \theta_l < \sigma_0$, without loss of generality, assume $a < b = \theta_l = \mu_{d_l+1}$. We claim that $\exists \varepsilon_i \ge 0$, $\varepsilon_i \to 0$, s.t.

$$a\left\|\left(v+w_i^*\right)^-\right\|_{L^2}^2 + b\left\|\left(v+w_i^*\right)^+\right\|_{L^2}^2 = (b-\varepsilon_i)\left\|v+w_i^*\right\|_{L^2}^2. \tag{5.15}$$

Otherwise, $\exists \alpha > 0$, $\varlimsup_{i\to+\infty} \varepsilon_i \ge \alpha$. Notice that $\|w_i^*\|_m \to +\infty \Rightarrow \|w_i^*\|_{L^2} \to +\infty$, consequently, (5.14) yields

$$M \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)\,v^2 + \frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla w_i^*\right|^2 + V(x)\left|w_i^*\right|^2$$
$$-\frac{a}{2}\left\|\left(v+w_i^*\right)^-\right\|_{L^2}^2 - \frac{b}{2}\left\|\left(v+w_i^*\right)^+\right\|_{L^2}^2$$
$$\geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)\,v^2 - \frac{1}{2}(b-\alpha)\|v\|_{L^2}^2 + \frac{\alpha}{2}\left\|w_i^*\right\|_{L^2}^2 \to +\infty. \tag{5.16}$$

No way! The claim is thus proved.

By (5.15) we get

$$(b-a-\varepsilon_i)\left\|\left(v+w_i^*\right)^-\right\|_{L^2}^2 = \varepsilon_i\left\|\left(v+w_i^*\right)^+\right\|_{L^2}^2. \tag{5.17}$$

Denote $\lambda_{l+1} = \mu_{d_l+1}$ with $j = \dim E(\lambda_{l+1})$, where $E(\lambda_{l+1})$ is the eigenspace spanned by λ_{l+1}. Set $w_i^* = t_{l+1}^{(i)}\psi_{l+1}^{(i)} + \widetilde{w}_i$, $t_{l+1}^{(i)} \geq 0$, $\psi_{l+1}^{(i)} \in E(\lambda_{l+1})$, $\left\|\psi_{l+1}^{(i)}\right\|_m = 1$, and $\widetilde{w}_i \in M_{l+1}$. As $\mu_{d_l+j+1} > b$, an argument analogous to (5.14) shows the boundedness of $\|\widetilde{w}_i\|_{L^2}$ via (5.15), and thus $t_{l+1}^{(i)} \to +\infty$. For the sake of this, $\exists I \in \mathbb{N}$, $\forall i \geq I$,

$$(b-a-\varepsilon_i)\left\|y_i^-\right\|_{L^2}^2 = \varepsilon_i\left\|y_i^+\right\|_{L^2}^2, \tag{5.18}$$

$y_i = \frac{v}{t_{l+1}^{(i)}} + \psi_{l+1}^{(i)} + \frac{\widetilde{w}_i}{t_{l+1}^{(i)}}$. Obviously, $y_i \to \psi_{l+1}^*$ in $H_m^1\left(\mathbb{R}^N\right)$, $\psi_{l+1}^* \in E(\lambda_{l+1})$, and hence $y_i \to \psi_{l+1}^*$ in $L^2\left(\mathbb{R}^N\right)$. Sending i to infinity for (5.18) obtains

$$\left\|\left(\psi_{l+1}^*\right)^-\right\|_{L^2} = 0. \tag{5.19}$$

This implies $\psi_{l+1}^* \geq 0$ a.e. on $\mathbb{R}^N$. On the other side, as V is a K–R potential, it follows from (V_2) that A defined as a sum of quadratic forms, has a nondegenerate strictly positive ground state, i.e., λ_1 is a simple eigenvalue and the corresponding eigenfunction ψ_1 is strictly positive (see Theorem XIII.48, [33]). Based on above argument, we derive a paradox that $\left\|\psi_{l+1}^*\right\|_m = 1$ with $\psi_{l+1}^* = 0$ a.e. on $\mathbb{R}^N$.

Coerciveness of $I(v+w,a,b)$ on M_l indicates the boundedness of any sequence $\{w_k\}_{k=1}^{+\infty} \subset M_l$, s.t.

$$I(v+w_k,a,b) \to F_{2l}(v,a,b). \tag{5.20}$$

Assume $w_k \rightharpoonup w_0$ in H_m^1. Notice that a convex set is weakly closed iff it is strongly closed. And $I(v+w,a,b)$ is convex in $w \in M_l$, so the weak lower semicontinuity of $I(v+w,a,b)$ on M_l is equivalent to the lower semicontinuity of $I(v+w,a,b)$ on M_l. Thanks to the coerciveness of $I(v+w,a,b)$ on M_l, it follows that

$$I(v+w_0,a,b) = F_{2l}(v,a,b). \tag{5.21}$$

Thereby, $\forall \widetilde{w} \in M_l$,

$$
\begin{aligned}
\left\langle I'(v+w_0,a,b), \widetilde{w}\right\rangle_m &= \int_{\mathbb{R}^N} \nabla w_0 \nabla \widetilde{w} + \int_{\mathbb{R}^N} V(x)\, w_0 \widetilde{w} \\
&\quad - a \int_{\mathbb{R}^N} (v+w_0)^- \cdot \widetilde{w} - b \int_{\mathbb{R}^N} (v+w_0)^+ \cdot \widetilde{w} \\
&= 0.
\end{aligned} \tag{5.22}
$$

Secondly, we claim that

$$
\left\| P_{M_l} I'(v+w_k,a,b)\right\|_{M_l} := \sup_{\substack{\varphi\in M_l \\ \|\varphi\|_m=1}} \left|\left\langle P_{M_l} I'(v+w_k,a,b), \varphi\right\rangle_m\right| \to 0. \tag{5.23}
$$

Indeed, for fixed $v \in N_l$, $\{w_k\}_{k=1}^{+\infty}$ is a minimizing sequence of the infimum $F_{2l}(v,a,b)$, and thus, using Ekeland variational principle, there exists a sequence $\{\widetilde{w}_k\}_{k=1}^{+\infty} \subset M_l$, with $\|\widetilde{w}_k - w_k\|_m \to 0$ and

$$
\left\| P_{M_l} I'(v+\widetilde{w}_k,a,b)\right\|_{M_l} \to 0. \tag{5.24}
$$

Observe that for $\varphi \in H_m^1\left(\mathbb{R}^N\right)$, $\|\varphi\|_m = 1$,

$$
\begin{aligned}
&\left\langle P_{M_l} I'(v+\widetilde{w}_k,a,b), \varphi\right\rangle_m \\
&= \int_{\mathbb{R}^N} \nabla P_{M_l} I'(v+\widetilde{w}_k,a,b)\, \nabla P_{M_l}\varphi + (V+m)\, P_{M_l} I'(v+\widetilde{w}_k,a,b)\cdot\varphi \\
&\quad + \int_{\mathbb{R}^N} \nabla P_{M_l} I'(v+\widetilde{w}_k,a,b)\, \nabla P_{N_l}\varphi.
\end{aligned} \tag{5.25}
$$

And by [22], it follows from $P_{M_l} I'(v+\widetilde{w}_k,a,b) \in H_m^1\left(\mathbb{R}^N\right)$ and $P_{N_l}\varphi \in H^2\left(\mathbb{R}^N\right)$ that $P_{M_l} I'(v+\widetilde{w}_k,a,b)\, \Delta P_{N_l}\varphi \in L^1\left(\mathbb{R}^N\right)$.

Hence,

$$
\begin{aligned}
&\left\langle P_{M_l} I'(v+\widetilde{w}_k,a,b), \varphi\right\rangle_m \\
&= \int_{\mathbb{R}^N} \nabla I'(v+\widetilde{w}_k,a,b)\, \nabla P_{M_l}\varphi + (V+m)\, I'(v+\widetilde{w}_k,a,b)\cdot P_{M_l}\varphi \\
&= \left\langle I'(v+\widetilde{w}_k,a,b), P_{M_l}\varphi\right\rangle_m .
\end{aligned} \tag{5.26}
$$

Consequently,

$$
\begin{aligned}
&\left\langle P_{M_l} I'(v+w_k,a,b) - P_{M_l} I'(v+\widetilde{w}_k,a,b), \varphi\right\rangle_m \\
&= \int_{\mathbb{R}^N} \nabla (w_k - \widetilde{w}_k)\, \nabla P_{M_l}\varphi + V(x)\,(w_k - \widetilde{w}_k)\, P_{M_l}\varphi
\end{aligned}
$$

$$-a\int_{\mathbb{R}^N}\left[(v+w_k)^- - (v+\widetilde{w}_k)^-\right]\cdot P_{M_l}\varphi$$

$$-b\int_{\mathbb{R}^N}\left[(v+w_k)^+ - (v+\widetilde{w}_k)^+\right]\cdot P_{M_l}\varphi. \tag{5.27}$$

Notice that

$$(v+w_k)^- - (v+\widetilde{w}_k)^- \le -(\widetilde{w}_k - w_k)^- \le |w_k - \widetilde{w}_k|, \tag{5.28}$$

$$(v+w_k)^- - (v+\widetilde{w}_k)^- \ge (w_k - \widetilde{w}_k)^- \ge -|w_k - \widetilde{w}_k|, \tag{5.29}$$

$$(v+w_k)^+ - (v+\widetilde{w}_k)^+ \le (w_k - \widetilde{w}_k)^+ \le |w_k - \widetilde{w}_k|, \tag{5.30}$$

$$(v+w_k)^+ - (v+\widetilde{w}_k)^+ \ge -(\widetilde{w}_k - w_k)^+ \ge -|w_k - \widetilde{w}_k|, \tag{5.31}$$

therefore, one deduces from (5.27) that for $\forall\varphi\in M_l$, $\|\varphi\|_m = 1$,

$$\begin{aligned}
&\left|\left\langle P_{M_l}I'(v+w_k,a,b),\varphi\right\rangle_m\right| \\
\le\ &\left|\left\langle P_{M_l}I'(v+w_k,a,b) - P_{M_l}I'(v+\widetilde{w}_k,a,b),\varphi\right\rangle_m\right| \\
&+\left|\left\langle P_{M_l}I'(v+\widetilde{w}_k,a,b),\varphi\right\rangle_m\right| \\
\le\ &[1+2(|a|+|b|+2m)]\cdot\|w_k-\widetilde{w}_k\|_m \\
&+\left\|P_{M_l}I'(v+\widetilde{w}_k,a,b)\right\|_{M_l} \\
\to\ &0,
\end{aligned} \tag{5.32}$$

alluding to (5.23), $\frac{1}{p}+\frac{1}{q}=1$, $p>\frac{N}{2}$ if $N\ge 4$, and $p=2$ if $N\le 3$.

By (5.23) we have

$$\begin{aligned}
&\left|\left\langle P_{M_l}I'(v+w_k,a,b),w_k-w_0\right\rangle_m\right| \\
=&\left|\left\langle I'(v+w_k,a,b),w_k-w_0\right\rangle_m\right|\to 0.
\end{aligned} \tag{5.33}$$

It might as well assume $a\le b$. In (5.22), taking $\widetilde{w}=w_k-w_0$ yields

$$\begin{aligned}
0\leftarrow&\left\langle I'(v+w_k,a,b)-I'(v+w_0,a,b),w_k-w_0\right\rangle_m \\
=&\int_{\mathbb{R}^N}|\nabla(w_k-w_0)|^2+V(x)(w_k-w_0)^2-a\|w_k-w_0\|_{L^2}^2 \\
&\quad-(b-a)\int_{\mathbb{R}^N}\left[(v+w_k)^+-(v+w_0)^+\right]\cdot(w_k-w_0) \\
\ge&\int_{\mathbb{R}^N}|\nabla(w_k-w_0)|^2+V(x)(w_k-w_0)^2-b\|w_k-w_0\|_{L^2}^2 \\
\ge&\frac{\theta_l-b}{\theta_l+m}\|w_k-w_0\|_m^2,
\end{aligned} \tag{5.34}$$

which implies that $w_k \to w_0$ in $H_m^1(\mathbb{R}^N)$ as $b < \theta_l$. If $b = \theta_l < \sigma_0$, set $w_k - w_0 = P_{E(\lambda_{l+1})}(w_k - w_0) + P_{M_{l+1}}(w_k - w_0)$, (5.34) obtains

$$
\begin{aligned}
0 \leftarrow & \left\langle I'(v + w_k, a, b) - I'(v + w_0, a, b), w_k - w_0 \right\rangle_m \\
\geq & \int_{\mathbb{R}^N} \left|\nabla P_{E(\lambda_{l+1})}(w_k - w_0)\right|^2 + V(x)\left|P_{E(\lambda_{l+1})}(w_k - w_0)\right|^2 \\
& -b\left\|P_{E(\lambda_{l+1})}(w_k - w_0)\right\|_{L^2}^2 \\
& + \int_{\mathbb{R}^N} \left|\nabla P_{M_{l+1}}(w_k - w_0)\right|^2 + V(x)\left|P_{M_{l+1}}(w_k - w_0)\right|^2 \\
& -b\left\|P_{M_{l+1}}(w_k - w_0)\right\|_{L^2}^2 \\
\geq & \left(\mu_{d_l+j+1} - b\right)\left\|P_{M_{l+1}}(w_k - w_0)\right\|_{L^2}^2,
\end{aligned}
\tag{5.35}
$$

and this gets

$$
P_{M_{l+1}} w_k \to P_{M_{l+1}} w_0 \text{ in } L^2\left(\mathbb{R}^N\right). \tag{5.36}
$$

Note that $w_k \rightharpoonup w_0$ in $H_m^1(\mathbb{R}^N) \Longleftrightarrow w_k \rightharpoonup w_0$ in $H^1(\mathbb{R}^N)$ (see Lemma 9.3), so we have

$$
\int_{\mathbb{R}^N} P_{E(\lambda_{l+1})} w_k \varphi \to \int_{\mathbb{R}^N} P_{E(\lambda_{l+1})} w_0 \varphi, \forall \varphi \in L^2\left(\mathbb{R}^N\right), \tag{5.37}
$$

and hence,

$$
P_{E(\lambda_{l+1})} w_k \to P_{E(\lambda_{l+1})} w_0 \text{ in } L^2\left(\mathbb{R}^N\right). \tag{5.38}
$$

(5.36) together with (5.38) derives

$$
w_k \to w_0 \text{ in } L^2\left(\mathbb{R}^N\right), \tag{5.39}
$$

and this shows $w_k \to w_0$ in $H_m^1(\mathbb{R}^N)$ by (5.34).

Finally, uniqueness follows directly from (4.19). The proof is complete. □

Set $I(v + \tau(v, a, b), a, b) = F_{2l}(v, a, b)$, $\tau(v, a, b) \in M_l$, we have

Lemma 5.2. *Under the hypothesis* (V^*), *if* $\Lambda(a, b) < \theta_l$, *then* $\tau(v, a, b)$ *is continuous on* $v \in N_l$.

Proof. Without loss of generality, assume $a \leq b$. Let $v_n \to v_0$ in N_l. Notice that

$$
\begin{aligned}
0 &= \left\langle I'\left(v_n+\tau\left(v_n,a,b\right),a,b\right)-I'\left(v_0+\tau\left(v_0,a,b\right),a,b\right),\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right\rangle_m \\
&= \int_{\mathbb{R}^N} \left|\nabla\left(\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right)\right|^2+V(x)\left|\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right|^2 \\
&\quad -a\left\|\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right\|_{L^2}^2 \\
&\quad -(b-a)\int_{\mathbb{R}^N}\left(v_n+\tau\left(v_n,a,b\right)\right)^+\cdot\left(\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right) \\
&\quad +(b-a)\int_{\mathbb{R}^N}\left(v_0+\tau\left(v_0,a,b\right)\right)^+\cdot\left(\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right),
\end{aligned} \tag{5.40}
$$

and

$$
\begin{aligned}
&\left(v_n+\tau\left(v_n,a,b\right)\right)^+-\left(v_0+\tau\left(v_0,a,b\right)\right)^+ \\
&\le \left(v_n-v_0+\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right)^+ \\
&\le \left(v_n-v_0\right)^++\left(\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right)^+,
\end{aligned} \tag{5.41}
$$

and also

$$
\begin{aligned}
&\left(v_n+\tau\left(v_n,a,b\right)\right)^+-\left(v_0+\tau\left(v_0,a,b\right)\right)^+ \\
&\ge -\left(v_0-v_n+\tau\left(v_0,a,b\right)-\tau\left(v_n,a,b\right)\right)^+ \\
&\ge \left(v_n-v_0\right)^-+\left(\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right)^-,
\end{aligned} \tag{5.42}
$$

we yield

$$
\begin{aligned}
0 &\ge \int_{\mathbb{R}^N} \left|\nabla\left(\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right)\right|^2+V(x)\left|\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right|^2 \\
&\quad -b\left\|\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right\|_{L^2}^2 \\
&\quad -(b-a)\int_{\mathbb{R}^N}\left(v_n-v_0\right)^+\cdot\left(\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right)^+ \\
&\quad -(b-a)\int_{\mathbb{R}^N}\left(v_n-v_0\right)^-\cdot\left(\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right)^- \\
&\ge \frac{\theta_l-b}{\theta_l+m}\left\|\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right\|_m^2 \\
&\quad -2(b-a)\left\|v_n-v_0\right\|_{L^2}\cdot\left\|\tau\left(v_n,a,b\right)-\tau\left(v_0,a,b\right)\right\|_{L^2},
\end{aligned} \tag{5.43}
$$

and thus

$$\begin{aligned}
&\frac{\theta_l - b}{\theta_l + m} \left\| \tau \left(v_n, a, b\right) - \tau \left(v_0, a, b\right) \right\|_m^2 \\
&\leq 2 \left(b - a\right) \left\| v_n - v_0 \right\| \cdot \left\| \tau \left(v_n, a, b\right) - \tau \left(v_0, a, b\right) \right\| \\
&\leq 4 \left(b - a\right) \left\| v_n - v_0 \right\|_m \cdot \left\| \tau \left(v_n, a, b\right) - \tau \left(v_0, a, b\right) \right\|_m ,
\end{aligned} \tag{5.44}$$

showing

$$\left\| \tau \left(v_n, a, b\right) - \tau \left(v_0, a, b\right) \right\|_m \leq 4 \left(b - a\right) \frac{\theta_l + m}{\theta_l - b} \left\| v_n - v_0 \right\|_m \to 0, \tag{5.45}$$

and consequently we arrive at the conclusion as desired. □

Proposition 5.3. *Under the hypothesis* (V^*), *for* $(a, b) \notin \Sigma(A)$, *if* $\Lambda(a, b) < \theta_l$ *or* $\Gamma(a, b) < \Lambda(a, b) = \theta_l < \sigma_0$, *then*

$$C_q(I, 0) \cong 0, \forall q \geq d_l + 1. \tag{5.46}$$

Proof. If $\Lambda(a, b) < \theta_l$, by Lemma 5.2, the proof is quite similar to that of the second half of Theorem 1.1(i) of [32]. In the case of $\Gamma(a, b) < \Lambda(a, b) = \theta_l < \sigma_0$, it is no harm in supposing $a < b$. Denote $I = I(a, b)$. As $(a, b) \notin \Sigma(A)$, an argument analogous to the proof of Theorem 3.4 shows that for $\varepsilon > 0$ suitably small and $\forall b^* \in [b - \varepsilon, b]$, $(a, b^*) \notin \Sigma(A)$ and thus $I(a, b^*)$ satisfies the (PS) condition. Therefore, we conclude (5.46) by Proposition 9.4 (see Appendix). The proof is complete. □

Lemma 5.4. *Under the hypothesis* (V^*), *let*

$$\left\langle I'\left(u_j, a, b\right), w\right\rangle_m = 0, \forall w \in M_l, j = 0, 1, \tag{5.47}$$

where $u_j = v + w_j$, $v \in N_l$, $w_j \in M_l$, $j = 0, 1$. *If*

(*i*) $\Lambda(a, b) < \theta_l$;

or

(*ii*) (V_2) *follows, and* $\Gamma(a, b) < \Lambda(a, b) = \theta_l < \sigma_0$, *and* $\exists j \in \{0, 1\}$, u_j *solves* (1.3),

then $w_1 = w_0$.

Proof. If $\Lambda(a, b) < \theta_l$, the conclusion follows directly from (4.19) and (4.20) so we focus on the second case. In view of (*ii*), we assume that (1.3) is solved by u_0, i.e.,

$$I(u_0, a, b) = 0, I'(u_0, a, b) = 0. \tag{5.48}$$

Observe that

$$\begin{aligned}
0 &= \left\langle I'(u_1, a, b) - I'(u_0, a, b), w\right\rangle_m \\
&= \int_{\mathbb{R}^N} |\nabla w|^2 + V(x) w^2 dx - a\left\langle u_1^- - u_0^-, w\right\rangle_{L^2} - b\left\langle u_1^+ - u_0^+, w\right\rangle_{L^2},
\end{aligned} \tag{5.49}$$

where $w = w_1 - w_0$, we have

$$\begin{aligned}&\left(b-\mu_{d_l+1}\right)\left\langle u_1^+-u_0^+,w\right\rangle_{L^2}+\left(a-\mu_{d_l+1}\right)\left\langle u_1^--u_0^-,w\right\rangle_{L^2}\\ &=\int_{\mathbb{R}^N}|\nabla w|^2+V(x)\,w^2dx-\mu_{d_l+1}\|w\|_{L^2}^2\geq 0.\end{aligned} \tag{5.50}$$

On the other hand, as $\left\langle u_1^+-u_0^+,w\right\rangle_{L^2}\geq 0$, $\left\langle u_1^--u_0^-,w\right\rangle_{L^2}\geq 0$, we obtain

$$\left(b-\mu_{d_l+1}\right)\left\langle u_1^+-u_0^+,w\right\rangle_{L^2}+\left(a-\mu_{d_l+1}\right)\left\langle u_1^--u_0^-,w\right\rangle_{L^2}\leq 0. \tag{5.51}$$

By employing (5.50) and (5.51)

$$\int_{\mathbb{R}^N}|\nabla w|^2+V(x)\,w^2dx=\mu_{d_l+1}\|w\|_{L^2}^2, \tag{5.52}$$

and then $w\in E\left(\mu_{d_l+1}\right)$.

Notice that $\Lambda(a,b)=\theta_l=\mu_{d_l+1}<\sigma_0$ and $a+b<2\theta_l$, there is no harm in assuming $a<b=\theta_l$. The combination of (5.50) and (5.51) leads to

$$\left\langle u_1^--u_0^-,w\right\rangle_{L^2}=0. \tag{5.53}$$

Define

$$\begin{aligned}\Omega_1&=\left\{x\in\mathbb{R}^N:u_0(x)\geq 0,u_1(x)\geq 0\right\};\\ \Omega_2&=\left\{x\in\mathbb{R}^N:u_0(x)\geq 0,u_1(x)\leq 0\right\};\\ \Omega_3&=\left\{x\in\mathbb{R}^N:u_0(x)\leq 0,u_1(x)\geq 0\right\};\\ \Omega_4&=\left\{x\in\mathbb{R}^N:u_0(x)\leq 0,u_1(x)\leq 0\right\}.\end{aligned}$$

Hence, (5.53) yields

$$\begin{aligned}0&=\left\langle u_1^--u_0^-,w\right\rangle_{L^2}\\ &=\int_{\Omega_2}u_1(u_1-u_0)-\int_{\Omega_3}u_0(u_1-u_0)+\int_{\Omega_4}(u_1-u_0)^2.\end{aligned} \tag{5.54}$$

Suppose to the contrary that $w\neq 0$. Notice that $\int_{\Omega_i}\left(u_1^--u_0^-\right)w\geq 0$, we deduce $|\Omega_4|=0$. Otherwise, $w=0$ a.e. on Ω_4. An argument analogous to (4.19) shows $w\in E\left(\mu_{d_l+1}\right)$. As $V\in L_{loc}^{\frac{N}{2}}\left(\mathbb{R}^N\right)$, by unique continuation theorem (see [17], [18], [30]), $w\equiv 0$ on $\mathbb{R}^N$ (there is an extensive literature on unique continuation and we refer the readers to [12], [13], [19], [39], [40], and so forth). However, this contradicts the hypothesis $w\neq 0$. So if $\left|\Omega_1^c\right|>0$, then $|\Omega_2\cup\Omega_3|>0$, $\Omega_1^c=\bigcup_{i=2}^{4}\Omega_i$. It might as well assume $|\Omega_2|>0$, $|\Omega_3|>0$. It follows that $u_1=0$

a.e. on Ω_2 and $u_0 = 0$ a.e. on Ω_3. Once again from unique continuation, $u_0 > 0$ a.e. on Ω_2, and $u_1 > 0$ a.e. on Ω_3.

Above argument indicates that $u_i \geq 0$ a.e. on $\mathbb{R}^N$ as $|\Omega_4| = 0$. Thus,

$$\left\langle I'(u_0, a, b), v\right\rangle_m = \int\limits_{\mathbb{R}^N} |\nabla v|^2 + V(x) v^2 dx - b\|v\|_{L^2}^2 = 0, \tag{5.55}$$

which alludes to $v = 0$. By (5.47),

$$\left\langle I'(u_0, a, b), w_0\right\rangle_m = \int\limits_{\mathbb{R}^N} |\nabla w_0|^2 + V(x)|w_0|^2 dx - b\|w_0\|_{L^2}^2 = 0, \tag{5.56}$$

$$\left\langle I'(u_1, a, b), w_1\right\rangle_m = \int\limits_{\mathbb{R}^N} |\nabla w_1|^2 + V(x)|w_1|^2 dx - b\|w_1\|_{L^2}^2 = 0, \tag{5.57}$$

and this yields $u_0, u_1 \in E\left(\mu_{d_l+1}\right)$, and consequently $u_i \perp \psi_1$. Since $u_0 = 0$ a.e. on Ω_3, and $u_1 = 0$ a.e. on Ω_2, unique continuation shows that $u_0 = u_1 = 0$ on $\mathbb{R}^N$, contradicting $w \neq 0$. □

Remark 5.5. It seems that the assumption analogous to "$\exists j \in \{0, 1\}$, u_j solves (1.3)" of Lemma 5.4(*ii*) is nonredundant for us in some cases. Indeed, for $v = \varphi_1$, $\|\varphi_1\|_{H_0^1} = 1$, we can always choose $t_0 > 0$ sufficiently small such that for $\forall t \in [0, t_0]$, $\varphi_1 + t\varphi_{l+1} \in \left(\widetilde{P}\right)^\circ$, where φ_1 and φ_l denote the eigenfunctions of the first eigenvalue $\widehat{\lambda}_1$ and the $l+1$-th eigenvalue $\widehat{\lambda}_{l+1}$ of $-\Delta$ with Dirichlet boundary condition and restricted on a bounded smooth domain $\Omega \subset \mathbb{R}^N$, and $\|\cdot\|_{H_0^1}$ represents the norm of $H_0^1(\Omega)$, and $\left(\widetilde{P}\right)^\circ$ is the interior of

$$\widetilde{P} = \text{closure}\left\{u \in C_0^1\left(\overline{\Omega}\right) : u > 0 \text{ on } \Omega,\, u|_{\partial\Omega} = 0,\, \frac{\partial u}{\partial \nu}|_{\partial\Omega} < 0\right\}.$$

Set $w_t = t\varphi_{l+1}$, and take $b = \widehat{\lambda}_{l+1}$, hence, for $v = \varphi_1$, $\forall \widetilde{w} \in \widehat{M}_l$,

$$\begin{aligned}\int\limits_{\Omega} \nabla(v + w_t)\nabla\widetilde{w} - a\int\limits_{\Omega}(v + w_t)^-\widetilde{w} - b\int\limits_{\Omega}(v + w_t)^+\widetilde{w}\\ = \int\limits_{\Omega}\nabla w_t\nabla\widetilde{w} - \widehat{\lambda}_{l+1}\int\limits_{\Omega} w_t\widetilde{w} = 0,\end{aligned} \tag{5.58}$$

where $\widehat{M}_l$ is the orthogonal complement in $H_0^1(\Omega)$ of $\widehat{N}_l$, and $\widehat{N}_l$ stands for the subspace spanned by the eigenfunctions corresponding to $\widehat{\lambda}_1, \cdots, \widehat{\lambda}_l$.

Lemma 5.6. *Under the hypothesis* (V^*), *assume* $\mu_{d_l+1} < \sigma_0$, *and* $m_{l+1}(a, b) < 0$. *If*

(*i*) $\Lambda(a, b) < \widetilde{\theta}_{l+1} := \Gamma\left(\mu_{d_{l+1}+1}, \sigma_0\right)$;

or

(*ii*) (V_2) *follows, and* $\Gamma(a, b) < \Lambda(a, b) = \widetilde{\theta}_{l+1} < \sigma_0$,

then $(a, b) \notin \Sigma(A)$. *In addition, if* $\lambda_l < \Gamma(a, b) \leq \Lambda(a, b) < \widetilde{\theta}_{l+1}$, *then* $C_q(I, 0) \cong \delta_{qd_{l+1}}G$.

Proof. By way of negation, there is a $u_0 = v_0 + w_0 \neq 0$, $v_0 \in N_{l+1}$, $w_0 \in M_{l+1}$, such that (1.3) holds. By Lemma 5.1 and Lemma 5.4,

$$F_{2l+1}(v_0, a, b) = I(v_0 + w_0, a, b) = 0. \tag{5.59}$$

If $v_0 \neq 0$, we may assume $\|v_0\|_m = 1$. This implies that $m_{l+1}(a, b) \geq 0$, contrary to assumption. If $v_0 = 0$, then $w_0 \neq 0$ and

$$I(u_0, a, b) = I(w_0, a, b) \geq \frac{\widetilde{\theta}_{l+1} - a}{2} \left\|w_0^-\right\|_{L^2}^2 + \frac{\widetilde{\theta}_{l+1} - b}{2} \left\|w_0^+\right\|_{L^2}^2. \tag{5.60}$$

If $\Lambda(a, b) < \widetilde{\theta}_{l+1}$, this will be positive, contradicting (5.59). If $\Lambda(a, b) = \widetilde{\theta}_{l+1}$, assume $b = \widetilde{\theta}_{l+1}$, then $w_0 \geq 0$ a.e. on $\mathbb{R}^N$, and thus $w_0 \perp \psi_1$. This shows $w_0 = 0$, violating $w_0 \neq 0$. The second half of Lemma 5.6 follows directly from Theorem 1.1(iv) of [32]. □

Lemma 5.7. *Under the hypothesis* (V^*), *if*

(i) $\Gamma(a, b) > \lambda_l$;

or

(ii) (V_2) *follows, and* $\lambda_1 < \Gamma(a, b) = \lambda_l < \Lambda(a, b)$,

then for each $w \in M_l$, $F_{1l}(w, a, b)$ *can be achieved and every maximizing sequence contains a convergent subsequence. Moreover, if* $\Gamma(a, b) > \lambda_l$, *the maximizer is unique.*

Proof. We first show that for fixed $w \in M_l$, $I(v + w, a, b)$ is anticoercive on N_l, i.e., for any sequence $\left\{v_j^*\right\} \in N_l$, $\left\|v_j^*\right\|_m \to +\infty$, $I(v_j^* + w, a, b) \to -\infty$. Otherwise, there is a renamed sequence $v_j^* \in N_l$, $\left\|v_j^*\right\|_m \to +\infty$, $I(v_j^* + w, a, b) \geq -\eta$, and $\eta > 0$.

(i) $\Gamma(a, b) > \lambda_l$. Thereby,

$$\begin{aligned} -\eta \leq & -\frac{\Gamma(a, b) - \lambda_l}{2(\lambda_l + m)} \left\|v_j^*\right\|_m^2 \\ & + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w|^2 + V(x)|w|^2 - \frac{\Gamma(a, b)}{2} \|w\|_{L^2}^2 \end{aligned} \tag{5.61}$$

and then derives the boundedness of $\left\|v_j^*\right\|_m$, violating the hypothesis.

(ii) $\lambda_1 < \Gamma(a, b) = \lambda_l < \Lambda(a, b)$. Assume $a = \lambda_l < b$. An argument analogous to (5.15) shows that $\exists \rho_j \geq 0$, $\rho_j \to 0$, s.t.

$$a \left\|\left(v_j^* + w\right)^-\right\|_{L^2}^2 + b \left\|\left(v_j^* + w\right)^+\right\|_{L^2}^2 = \left(a + \rho_j\right) \left\|v_j^* + w\right\|_{L^2}^2. \tag{5.62}$$

Set $v_j^* = \widetilde{v}_j + t_l^{(j)} \psi_l^{(j)}$, $t_l^{(j)} \geq 0$, $\psi_l^{(j)} \in E(\lambda_l)$, $\left\|\psi_l^{(j)}\right\|_m = 1$, and $\widetilde{v}_j \in N_{l-1}$. Denote $m_0 = \dim E(\lambda_l)$. Observe that $\mu_{d_l - m_0} < a$ as $l \geq 2$, it follows that $\left\|\widetilde{v}_j\right\|_{L^2}$ is bounded, and thus $t_l^{(j)} \to +\infty$. On this account, $\exists J \in \mathbb{N}$, $\forall j \geq J$,

$$\left(b-a-\rho_j\right)\left\|z_j^+\right\|_{L^2}^2=\rho_j\left\|z_j^-\right\|_{L^2}^2, \tag{5.63}$$

$z_j=\frac{\widetilde{v}_j}{t_l^{(j)}}+\psi_l^{(j)}+\frac{w}{t_l^{(j)}}$. Clearly, $z_j\to\psi_l^*$ in $H_m^1\left(\mathbb{R}^N\right)$, $\psi_l^*\in E\left(\lambda_l\right)$, and hence $z_j\to\psi_l^*$ in $L^2\left(\mathbb{R}^N\right)$. Let $j\to+\infty$, (5.63) yields

$$\left\|\left(\psi_l^*\right)^+\right\|_{L^2}=0. \tag{5.64}$$

This implies $\psi_l^*\le 0$ a.e. on $\mathbb{R}^N$, contradicting the fact that ψ_1 has a constant sign on $\mathbb{R}^N$.

Anticoerciveness of $I\left(v+w,a,b\right)$ on N_l indicates that any sequence $\left\{v_j^*\right\}\subset N_l$ such that

$$I\left(v_j^*+w,a,b\right)\to F_{1l}\left(w,a,b\right), \tag{5.65}$$

$\left\|v_j^*\right\|_m$ is bounded, and $\dim N_l<+\infty$ alludes to the convergence of subsequence. Uniqueness comes from Lemma 4.9, ending the proof. □

Set $I\left(\theta\left(w,a,b\right)+w,a,b\right)=F_{1l}\left(w,a,b\right)$, $\theta\left(w,a,b\right)\in N_l$, we have

Lemma 5.8. *Under the hypothesis* (V^*), *if* $\Gamma\left(a,b\right)>\lambda_l$, *then* $\theta\left(w,a,b\right)$ *is continuous on* $w\in M_l$.

Proof. Without loss of generality, assume $a\le b$. Let $w_n\to w_0$ in M_l. Observe that

$$\begin{aligned}0&=\left\langle I'\left(\theta\left(w_n,a,b\right)+w_n,a,b\right)-I'\left(\theta\left(w_0,a,b\right)+w_0,a,b\right),\theta\left(w_n,a,b\right)-\theta\left(w_0,a,b\right)\right\rangle_m\\&=\int_{\mathbb{R}^N}\left|\nabla\left(\theta\left(w_n,a,b\right)-\theta\left(w_0,a,b\right)\right)\right|^2+V(x)\left|\theta\left(w_n,a,b\right)-\theta\left(w_0,a,b\right)\right|^2\\&\quad-a\left\|\theta\left(w_n,a,b\right)-\theta\left(w_0,a,b\right)\right\|_{L^2}^2\\&\quad-(b-a)\int_{\mathbb{R}^N}\left[\left(\theta\left(w_n,a,b\right)+w_n\right)^+-\left(\theta\left(w_0,a,b\right)+w_0\right)^+\right]\\&\quad\cdot\left(\theta\left(w_n,a,b\right)-\theta\left(w_0,a,b\right)\right).\end{aligned} \tag{5.66}$$

By

$$\begin{aligned}&\left(\theta\left(w_n,a,b\right)+w_n\right)^+-\left(\theta\left(w_0,a,b\right)+w_0\right)^+\\&\ge\left(\theta\left(w_n,a,b\right)-\theta\left(w_0,a,b\right)\right)^-+\left(w_n-w_0\right)^-,\end{aligned} \tag{5.67}$$

and

$$\begin{aligned}&\left(\theta\left(w_n,a,b\right)+w_n\right)^+-\left(\theta\left(w_0,a,b\right)+w_0\right)^+\\&\le\left(\theta\left(w_n,a,b\right)-\theta\left(w_0,a,b\right)\right)^++\left(w_n-w_0\right)^+,\end{aligned} \tag{5.68}$$

therefore, (5.66) is reduced to

$$
\begin{aligned}
&\frac{a-\lambda_l}{\lambda_l+m}\|\theta(w_n,a,b)-\theta(w_0,a,b)\|_m^2 \\
&\le -(b-a)\int\limits_{\mathbb{R}^N}(w_n-w_0)^-\cdot(\theta(w_n,a,b)-\theta(w_0,a,b))^+ \\
&\quad -(b-a)\int\limits_{\mathbb{R}^N}(w_n-w_0)^+\cdot(\theta(w_n,a,b)-\theta(w_0,a,b))^- \\
&\le 4(b-a)\|w_n-w_0\|_m\cdot\|\theta(w_n,a,b)-\theta(w_0,a,b)\|_m,
\end{aligned}
\tag{5.69}
$$

and further derives

$$
\|\theta(w_n,a,b)-\theta(w_0,a,b)\|_m \le 4(b-a)\frac{\lambda_l+m}{a-\lambda_l}\|w_n-w_0\|_m \to 0. \tag{5.70}
$$

The assertion follows. □

Proposition 5.9. *Under the hypotheses of Lemma 5.7, if $\Lambda(a,b)<\sigma_0$, $(a,b)\notin\Sigma(A)$, $\Gamma(a,b)>\lambda_l$ or $\lambda_1<\Gamma(a,b)=\lambda_l<\Lambda(a,b)$, then*

$$
C_q(I,0)\cong 0,\forall q\le d_l-1. \tag{5.71}
$$

Proof. The combination of $\Lambda(a,b)<\sigma_0$ and $(a,b)\notin\Sigma(A)$ determines the compactness of I in terms of Theorem 3.4. If $\Gamma(a,b)>\lambda_l$, by Lemma 5.7 and Lemma 5.8, (5.71) follows directly from the proof of the first half of Theorem 1.1(i) of [32]. An argument analogous to the proof of Proposition 5.3 deals with the case $\lambda_1<\Gamma(a,b)=\lambda_l<\Lambda(a,b)$. □

Lemma 5.10. *Under the hypothesis (V^*), let*

$$
\left\langle I'\left(u_j,a,b\right),v\right\rangle_m=0,\forall v\in N_l,\ j=0,1, \tag{5.72}
$$

where $u_j=v_j+w$, $w\in M_l$, $v_j\in N_l$, $j=0,1$. If
(i) $\Gamma(a,b)>\lambda_l$;
or
(ii) (V_2) follows, and $\lambda_1<\Gamma(a,b)=\lambda_l<\Lambda(a,b)$, and $\exists j\in\{0,1\}$, u_j solves (1.3),
then $v_1=v_0$.

Proof. Mimicking the proof of Lemma 4.9 and Lemma 5.4 we arrive at the statement (i) and (ii) respectively.

Lemma 5.11. *Under the hypotheses of Lemma 5.7, if $M_l(a,b)>0$, then $(a,b)\notin\Sigma(A)$. In addition, if $\mu_{d_l+1}<\sigma_0$ and $\lambda_l<\Gamma(a,b)\le\Lambda(a,b)<\widetilde{\theta}_{l+1}$, then $C_q(I,0)\cong\delta_{qd_l}G$.*

Proof. Suppose to the contrary that $\exists u_0=v_0+w_0\ne 0$, $v_0\in N_l$, $w_0\in M_l$, such that u_0 solves (1.3). Evidently, $w_0\ne 0$. Set $\|w_0\|_m=1$, by Lemma 5.10,

$$F_{1l}(w_0,a,b)=I(v_0+w_0,a,b)=0, \tag{5.73}$$

violating $M_l(a,b)>0$. The second half of Lemma 5.11 originates from the proof of Theorem 1.1(ii) of [32]. □

Lemma 5.12. *Under the hypothesis* (V^*), *assume* $\mu_{d_l+1}<\sigma_0$, *if*
(1) $\mu_{d_{l+1}}<\Gamma(a,b)\le\Lambda(a,b)<\widetilde{\theta}_{l+1}$;
or
(2) (V_2) *follows, and* $\mu_{d_{l+1}}=\Gamma(a,b)<\Lambda(a,b)<\widetilde{\theta}_{l+1}$,
then (*i*) $M_{l+1}(a,b)>0$; (*ii*) $(a,b)\notin\Sigma(A)$; (*iii*) $C_q(I,0)\cong\delta_{qd_{l+1}}G$.

Proof. Notice that $\forall w\in M_{l+1}$, $\|w\|_m=1$,

$$\begin{aligned}F_{1l+1}(w,a,b)&=\sup_{v\in N_{l+1}} I(v+w,a,b)\\&\ge I(w,a,b)\\&\ge\frac{\widetilde{\theta}_{l+1}-\Lambda(a,b)}{2}\|w\|_{L^2}^2>0.\end{aligned} \tag{5.74}$$

We deduce from (5.74) that

$$M_{l+1}(a,b)>0 \tag{5.75}$$

and this gets (*i*). An argument analogous to that of Lemma 5.11 derives (*ii*). (*iii*) still follows from the proof of Theorem 1.1(ii) of [32] for the case (1) while the identical consequence is concluded by Proposition 9.4 with respect to the case (2). □

Proposition 5.13. *Suppose* (V_2). *Under the hypothesis* (V^*), *if* $\mu_{d_l+1}<\sigma_0$ *and* $\mu_{d_{l+1}+1}<\sigma_0$, $\mu_{d_{l+1}}<\Gamma(a,b)<\Lambda(a,b)=\mu_{d_{l+1}+1}$, *then* (*i*) $M_{l+1}(a,b)>0$; (*ii*) $(a,b)\notin\Sigma(A)$; (*iii*) $C_q(I,0)\cong\delta_{qd_{l+1}}G$.

Proof. Without loss of generality, assume $\mu_{d_{l+1}}<a<b=\mu_{d_{l+1}+1}$. Clearly, (5.74) shows $M_{l+1}(a,b)\ge 0$. Argue by contradiction, $M_{l+1}(a,b)=0$. As $\mu_{d_{l+1}+1}$ is an eigenvalue of A, set $\mu_{d_{l+2}}=\mu_{d_{l+1}+1}$, by Lemma 5.16 below, $\exists w\in M_{l+1}$, $\|w\|_m=1$,

$$\begin{aligned}0&=M_{l+1}\left(a,\mu_{d_{l+2}}\right)\\&=I\left(\theta\left(w,a,\mu_{d_{l+2}}\right)+w,a,\mu_{d_{l+2}}\right)\\&\ge I\left(\theta\left(w,\mu_{d_{l+2}},\mu_{d_{l+2}}\right)+w,a,\mu_{d_{l+2}}\right).\end{aligned} \tag{5.76}$$

Set $\zeta=\theta\left(w,\mu_{d_{l+2}},\mu_{d_{l+2}}\right)$. We now claim that

$$I\left(\zeta+w,a,\mu_{d_{l+2}}\right)>I\left(\zeta+w,\mu_{d_{l+2}},\mu_{d_{l+2}}\right). \tag{5.77}$$

If not, we yield

$$\left\|(\zeta+w)^-\right\|_{L^2}=0 \tag{5.78}$$

and derive $\zeta \neq 0$ accordingly. Observe that $\left\langle I'\left(\zeta + w, \mu_{d_{l+2}}, \mu_{d_{l+2}}\right), \widetilde{v}\right\rangle_m = 0$ for $\forall \widetilde{v} \in N_{l+1}$, we obtain

$$\mu_{d_{l+1}} \|\zeta\|_{L^2}^2 \geq \int_{\mathbb{R}^N} |\nabla \zeta|^2 + V(x)\zeta^2 = \mu_{d_{l+2}} \|\zeta\|_{L^2}^2, \tag{5.79}$$

and this indicates $\zeta = 0$, conflicting with $\zeta \neq 0$. The claim is thus proved.

Consequently, a self-contradictory inequality

$$0 = M_{l+1}\left(a, \mu_{d_{l+2}}\right) > I\left(\zeta + w, \mu_{d_{l+2}}, \mu_{d_{l+2}}\right) = I\left(w, \mu_{d_{l+2}}, \mu_{d_{l+2}}\right) \geq 0 \tag{5.80}$$

arises from the combination of (5.76) and (5.77). That's the precise statement (i).

A standard argument gets (ii). Notice that $\mu_{d_{l+1}+1} < \sigma_0$ via Theorem 3.4 alludes to the (PS) condition, by Proposition 9.4 and Lemma 5.12(ii), we arrive at (iii). The assertion follows. □

Lemma 5.14. *Under the hypotheses of Lemma 5.7, if* $\Lambda(a,b) < \theta_l$, *then* (1) $m_l(a,b) < 0$; (2) $(a,b) \notin \Sigma(A)$; (3) $C_q(I,0) \cong \delta_{qd_l} G$.

Proof. Observe that (2) and (3) directly follow from (1) in view of Lemma 5.6 and Proposition 9.4, so we focus on (1). If $\Gamma(a,b) > \lambda_l$, according to the definition, for $\forall v \in N_l$, $\|v\|_m = 1$,

$$\begin{aligned} I(v,a,b) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + V(x)v^2 - \frac{a}{2}\left\|v^-\right\|_{L^2}^2 - \frac{b}{2}\left\|v^+\right\|_{L^2}^2 \\ &\leq \frac{\lambda_l - \Gamma(a,b)}{2}\|v\|_{L^2}^2 < 0, \end{aligned} \tag{5.81}$$

and hence $F_{2l}(v,a,b) < 0$. We can always choose $\{v_n\}_{n=1}^{+\infty} \subset N_l$, $\|v_n\|_m = 1$, s.t.

$$F_{2l}(v_n,a,b) \to m_l(a,b). \tag{5.82}$$

As $\dim N_l < +\infty$, there exists a renamed subsequence $\{v_n\}_{n=1}^{+\infty}$, $v_n \to v_0 \in N_l$, $\|v_0\|_m = 1$. Using Lemma 5.2, we have

$$F_{2l}(v_0,a,b) = m_l(a,b). \tag{5.83}$$

If $\Gamma(a,b) = \lambda_l$, it is no harm making the hypothesis $a = \lambda_l < b$. We show that (5.81) follows for $\forall v \in N_l$, $\|v\|_m = 1$. Otherwise, $\exists v^* \in N_l$, $\|v^*\|_m = 1$,

$$0 = I\left(v^*,a,b\right) \leq \frac{1}{2}\int_{\mathbb{R}^N} \left|\nabla v^*\right|^2 + V(x)\left|v^*\right|^2 - \frac{\lambda_l}{2}\left\|v^*\right\|_{L^2}^2 \leq 0. \tag{5.84}$$

Therefore, (5.84) yields

$$\int_{\mathbb{R}^N} \left|\nabla v^*\right|^2 + V(x)\left|v^*\right|^2 - \lambda_l\left\|v^*\right\|_{L^2}^2 = 0 \tag{5.85}$$

and $\left\|(v^*)^+\right\|_{L^2} = 0$, indicating that v^* is an eigenvector of λ_l with $v^* \leq 0$ a.e. on $\mathbb{R}^N$. Since $v^* \perp \psi_1$, we get $v^* = 0$ a.e. on $\mathbb{R}^N$, contradicting $v^* \neq 0$. The lemma is thus proved. □

Proposition 5.15. *Suppose* (V_2). *Under the hypothesis* (V^*), *if* $\lambda_l < \Gamma(a,b) < \Lambda(a,b) = \theta_l < \sigma_0$, *then* (*i*) $m_l(a,b) < 0$; (*ii*) $(a,b) \notin \Sigma(A)$; (*iii*) $C_q(I,0) \cong \delta_{qd_l} G$.

Proof. For fixed $v \in N_l$, set $\|v\|_m = 1$, we obtain

$$I(v,a,b) \leq -\frac{a-\lambda_l}{2(\lambda_l + m)}, \tag{5.86}$$

and thus

$$m_l(a,b) \leq -\frac{a-\lambda_l}{2(\lambda_l + m)}. \tag{5.87}$$

By employing Lemma 5.6 we get (*ii*). Due to $\theta_l < \sigma_0$, (*ii*) derives the (PS) condition. The combination of Proposition 9.4 and Lemma 5.14 determines (*iii*). □

5.2. Minimal and maximal curves

In this section we construct the lower and the upper curves emanating from (λ_l, λ_l). Aiming at this problems, we furnish some lemmas adapted to our needs. For the sake of convenience, in what follows, we assume:

(V_0) Let V be a K–R potential and suppose (V_2) holds, $\sigma_{\text{dis}}(A) \neq \varnothing$, $\inf \sigma(A) = \inf \sigma_{\text{dis}}(A)$, $\{\lambda_i\}_{i=1}^{l+1} \in \sigma_{\text{dis}}(A)$, $\sigma_{\text{dis}}(A) \cap [\mu_1, \lambda_{l+1}] = \{\lambda_i\}_{i=1}^{l+1}$, $l \geq 2$, s.t., $\lambda_1 = \mu_1 = \inf \sigma(A) < \lambda_2 < \cdots < \lambda_{l+1} < \sigma_0$.

Define

$$Q^*_{l+1} = \left\{(a,b) \in \mathbb{R}^2 : \lambda_l \leq \Gamma(a,b) \leq \Lambda(a,b) \leq \widetilde{\theta}_{l+1} = \Gamma\left(\mu_{d_{l+1}+1}, \sigma_0\right)\right\};$$
$$B^0_M = \left\{(a,b) \in Q^*_{l+1} : M_l(a,b) = 0\right\};$$
$$B^0_m = \left\{(a,b) \in Q^*_{l+1} : m_{l+1}(a,b) = 0\right\};$$
$$B^* = \left\{(a,b) \in Q^*_{l+1} : \lambda_l \leq a < \lambda_{l+1} < b \leq \widetilde{\theta}_{l+1}\right\};$$
$$B_* = \left\{(a,b) \in Q^*_{l+1} : \lambda_l \leq b < \lambda_{l+1} < a \leq \widetilde{\theta}_{l+1}\right\};$$
$$\widetilde{D} = \left\{(a,b) \in \mathbb{R}^2 : \Gamma(a,b) \geq \lambda_l\right\};$$
$$D = \left\{(a,b) \in \mathbb{R}^2 : \lambda_l \leq \Gamma(a,b) \leq \Lambda(a,b) < \sigma_0\right\};$$
$$D^+_M = \{(a,b) \in D : M_l(a,b) > 0\};$$
$$D^0_M = \{(a,b) \in D : M_l(a,b) = 0\};$$
$$D^-_M = \{(a,b) \in D : M_l(a,b) < 0\};$$
$$F = \left\{(a,b) \in \mathbb{R}^2 : -\infty < \Gamma(a,b) \leq \Lambda(a,b) \leq \widetilde{\theta}_{l+1}\right\};$$

$$F_m^+ = \{(a,b) \in F : m_{l+1}(a,b) > 0\};$$
$$F_m^0 = \{(a,b) \in F : m_{l+1}(a,b) = 0\};$$
$$F_m^- = \{(a,b) \in F : m_{l+1}(a,b) < 0\}.$$

Without the hypothesis of compact embedding, the following lemma considerably improves the conclusion of Lemma 3.8 of [38] (see also Lemma 7.3.8 of [36]):

Lemma 5.16. *Under the hypothesis* (V_0)*, if* $\Lambda(a,b) < \sigma_0$*,* $M_l(a,b) \le 0$*, then there is a* $w_0 \in M_l$*,* $\|w_0\|_m = 1$*, s.t.*

$$F_{1l}(w_0, a, b) = M_l(a,b). \tag{5.88}$$

Proof. Without loss of generality, assume $\Lambda(a,b) > \mu_1$. Suppose $\sigma_{\text{dis}}(A) \cap [\mu_1, \Lambda(a,b)] = \{\lambda_i\}_{i=1}^s$, $s \ge l+1$, and let E_{s-l} be the eigenspace spanned by $\lambda_{l+1}, \cdots, \lambda_s$. In view of (5.5), there is a sequence $\{w_k\}_{k=1}^{+\infty} \subset M_l$, s.t., $\|w_k\|_m = 1$, and

$$F_{1l}(w_k, a, b) \to M_l(a,b). \tag{5.89}$$

Let $w_k \rightharpoonup w_0$ in $H_m^1(\mathbb{R}^N)$. We claim $w_0 \ne 0$. Otherwise, set $w_k = P_{E_{s-l}} w_k + P_{M_s} w_k$, then we obtain $P_{E_{s-l}} w_k \to 0$ in $L^2(\mathbb{R}^N)$.

Observe that

$$F_{1l}(w_k, a, b) \ge \frac{1}{2} \int_{\mathbb{R}^N} |\nabla w_k|^2 + V w_k^2 - \frac{a}{2} \|w_k^-\|_{L^2}^2 - \frac{b}{2} \|w_k^+\|_{L^2}^2, \tag{5.90}$$

hence,

$$\begin{aligned} 2F_{1l}(w_k, a, b) \ge (\lambda_{l+1} - \Lambda(a,b)) \cdot \left\| P_{E_{s-l}} w_k \right\|_{L^2}^2 \\ + \left(\widetilde{\theta}_s - \Lambda(a,b)\right) \cdot \left\| P_{M_s} w_k \right\|_{L^2}^2, \end{aligned} \tag{5.91}$$

where $\widetilde{\theta}_s = \Gamma\left(\mu_{d_s+1}, \sigma_0\right)$. If $M_l(a,b) < 0$, (5.91) is self-contradictory as $\left\| P_{E_{s-l}} w_k \right\|_{L^2} \to 0$. If $M_l(a,b) = 0$, (5.91) gets

$$\left\| P_{M_s} w_k \right\|_{L^2} \to 0, \tag{5.92}$$

indicating $\|w_k\|_{L^2} \to 0$. Consequently, the contradiction manifests by

$$1 = \|w_k\|_m^2 \le (\Lambda(a,b) + m) \|w_k\|_{L^2}^2 + 2F_{1l}(w_k, a, b) \to 0 \tag{5.93}$$

and the claim is verified accordingly.

For fixed $k \in \mathbb{N}$, $\forall v \in N_l$, we have

$$I(v + w_k, a, b) \le F_{1l}(w_k, a, b). \tag{5.94}$$

Notice that for fixed $v \in N_l$, I is weakly lower semicontinuous on M_l, and thereby,

$$I\left(v+w_0,a,b\right)\le M_l\left(a,b\right). \tag{5.95}$$

Set $\widetilde{v}=\frac{v}{\|w_0\|_m}$, $\widetilde{w}_0=\frac{w_0}{\|w_0\|_m}$. Divide (5.95) by $\|w_0\|_m^2$,

$$I\left(\widetilde{v}+\widetilde{w}_0,a,b\right)\le\frac{M_l\left(a,b\right)}{\|w_0\|_m^2}, \tag{5.96}$$

yielding

$$M_l\left(a,b\right)\le F_{1l}\left(\widetilde{w}_0,a,b\right)\le\frac{M_l\left(a,b\right)}{\|w_0\|_m^2}, \tag{5.97}$$

and this shows that if $M_l\left(a,b\right)<0$, then $\|w_0\|_m=1$, alluding to $w_k\to w_0$ in $H_m^1\left(\mathbb{R}^N\right)$. We conclude the proof as required. □

Lemma 5.17. *Under the hypotheses of Lemma 5.16, if $(a,b)\in B_M^0\backslash\left(\lambda_{l+1},\lambda_{l+1}\right)$, then $(a,b)\in B^*$ or $(a,b)\in B_*$.*

Proof. For fixed $(a_0,b_0)\in Q_{l+1}^*\backslash\left(B^*\cup B_*\right)$, if $a_0\neq b_0$, then $\left(\Gamma\left(a_0,b_0\right),\Lambda\left(a_0,b_0\right)\right)\cap\sigma\left(A\right)=\varnothing$. Let $(a_0,b_0)\neq\left(\lambda_{l+1},\lambda_{l+1}\right)$. We are confronted with the following cases:

(1) $\lambda_l\le\Gamma\left(a_0,b_0\right)\le\Lambda\left(a_0,b_0\right)\le\lambda_{l+1}$.

(*i*) $\Lambda\left(a_0,b_0\right)<\lambda_{l+1}$.

Observe that for $\varphi\in E\left(\lambda_{l+1}\right)$, $\widetilde{w}\in M_{l+1}$, $\|\varphi+\widetilde{w}\|_m=1$,

$$\begin{aligned}I\left(\varphi+\widetilde{w},a_0,b_0\right)&=\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\varphi|^2+V\left(x\right)\varphi^2+\frac{1}{2}\int_{\mathbb{R}^N}|\nabla\widetilde{w}|^2+V\left(x\right)\widetilde{w}^2\\&\quad-\frac{a_0}{2}\int_{\mathbb{R}^N}\left|\left(\varphi+\widetilde{w}\right)^-\right|^2-\frac{b_0}{2}\int_{\mathbb{R}^N}\left|\left(\varphi+\widetilde{w}\right)^+\right|^2\\&\ge\frac{1}{2}\left[\lambda_{l+1}-\Lambda\left(a_0,b_0\right)\right]\cdot\|\varphi\|_{L^2}^2+\frac{1}{2}\left[\widetilde{\theta}_{l+1}-\Lambda\left(a_0,b_0\right)\right]\cdot\|\widetilde{w}\|_{L^2}^2\\&>0.\end{aligned} \tag{5.98}$$

One deduces from (5.98) that

$$M_l\left(a_0,b_0\right)\ge 0. \tag{5.99}$$

If $M_l\left(a_0,b_0\right)=0$, by Lemma 5.16, $\exists w_0\in M_l$, $\|w_0\|_m=1$, $F_{1l}\left(w_0,a_0,b_0\right)=0$, contradicting (5.98).

(*ii*) $\Lambda\left(a_0,b_0\right)=\lambda_{l+1}$.

In case that $M_l\left(a_0,b_0\right)=0$, again by Lemma 5.16, $\exists w_0^*\in M_l$, $\left\|w_0^*\right\|_m=1$, $F_{1l}(w_0^*,a_0,b_0)=0$. Set $w_0^*=\varphi_0^*+\widetilde{w}_0^*$, $\varphi_0^*\in E\left(\lambda_{l+1}\right)$, $\widetilde{w}_0^*\in M_{l+1}$. Evidently, $\widetilde{w}_0^*=0$. Therefore, the contradiction manifests by

$$0\ge I\left(\varphi_0^*,a_0,b_0\right)=\frac{\lambda_{l+1}}{2}\left\|\varphi_0^*\right\|_{L^2}^2-\frac{a_0}{2}\left\|\left(\varphi_0^*\right)^-\right\|_{L^2}^2-\frac{b_0}{2}\left\|\left(\varphi_0^*\right)^+\right\|_{L^2}^2>0 \tag{5.100}$$

due to the facts $a_0<\lambda_{l+1}$ and $\varphi_0^*\perp\psi_1$.

(2) $\lambda_{l+1} \leq \Gamma(a_0, b_0) \leq \Lambda(a_0, b_0) \leq \widetilde{\theta}_{l+1}$.

Take account of the case $\Lambda(a_0, b_0) < \widetilde{\theta}_{l+1}$. By Lemma 5.12, we get

$$M_{l+1}(a_0, b_0) > 0, \tag{5.101}$$

and

$$C_q\left(I_{(a_0,b_0)}, 0\right) \cong \delta_{qd_{l+1}} G, \tag{5.102}$$

where $I_{(a_0,b_0)}(u) = I(u, a_0, b_0)$.

We claim $M_l(a_0, b_0) < 0$, or else invoking Lemma 4.9, Corollary 4.10, and Theorem 1.1(ii) of [32], we obtain

$$C_q\left(I_{(a_0,b_0)}, 0\right) \cong \delta_{qd_l} G, \tag{5.103}$$

contradicting (5.102). The claim is proved.

Consequently,

$$M_l\left(a_0, \widetilde{\theta}_{l+1}\right) \leq M_l(a_0, b_0) < 0 \tag{5.104}$$

if $a_0 \leq b_0 < \widetilde{\theta}_{l+1}$ and

$$M_l\left(\widetilde{\theta}_{l+1}, b_0\right) \leq M_l(a_0, b_0) < 0 \tag{5.105}$$

if $b_0 \leq a_0 < \widetilde{\theta}_{l+1}$, and thus $M_l\left(\widetilde{\theta}_{l+1}, \widetilde{\theta}_{l+1}\right) < 0$. The conclusion follows. □

Lemma 5.18. *Under the hypotheses of Lemma 5.16, if $(a, b) \in D_M^0$, $\Gamma(a, b) > \lambda_l$ or $\lambda_l = a < \lambda_{l+1} < b$, then for $\forall b_1 \in [\lambda_l, b)$, $\forall b_2 \in (b, \sigma_0)$, $(a, b_1) \in D_M^+$, $(a, b_2) \in D_M^-$.*

Proof. Notice that for $\forall \widetilde{b} < b$,

$$\begin{aligned} M_l\left(a, \widetilde{b}\right) &= \inf_{w\in M_l, \|w\|_m=1} F_{1l}\left(w, a, \widetilde{b}\right) \\ &= \inf_{w\in M_l, \|w\|_m=1} \sup_{v\in N_l} I\left(v + w, a, \widetilde{b}\right) \\ &\geq \inf_{w\in M_l, \|w\|_m=1} \sup_{v\in N_l} I(v + w, a, b) = M_l(a, b). \end{aligned} \tag{5.106}$$

By way of negation, $\exists \widetilde{b} \in [\lambda_l, b)$, $M_l\left(a, \widetilde{b}\right) = 0$. Based on Lemma 5.7 and Lemma 5.16, $\exists \widetilde{w} \in M_l$, $\|\widetilde{w}\|_m = 1$, s.t.

$$I\left(\theta\left(\widetilde{w}, a, \widetilde{b}\right) + \widetilde{w}, a, \widetilde{b}\right) = F_{1l}\left(\widetilde{w}, a, \widetilde{b}\right) = M_l\left(a, \widetilde{b}\right). \tag{5.107}$$

A standard argument yields

$$I\left(\theta(\widetilde{w}, a, b) + \widetilde{w}, a, \widetilde{b}\right) > I(\theta(\widetilde{w}, a, b) + \widetilde{w}, a, b). \tag{5.108}$$

Therefore, the absurd assertion occurs as

$$0 = F_{1l}\left(\widetilde{w}, a, \widetilde{b}\right) > F_{1l}\left(\widetilde{w}, a, b\right) \geq M_l\left(a, b\right) = 0. \tag{5.109}$$

Likewise, we get the second half of Lemma 5.18. □

Lemma 5.19. *Under the hypothesis* (V_0)*, if* $M_l(a,b) > 0$*,* $\Lambda(a,b) < \sigma_0$*, then there exists* $\widetilde{\varepsilon}_0 > 0$*,* $\forall \widetilde{b} \in [b - \widetilde{\varepsilon}_0, b + \widetilde{\varepsilon}_0]$*,* $M_l\left(a, \widetilde{b}\right) > 0$*.*

Proof. Suppose to the contrary that $\exists \widetilde{b}_k \in \mathbb{R}$, $\widetilde{b}_k \to b$, $M_l\left(a, \widetilde{b}_k\right) \leq 0$. Employing Lemma 5.16, $\exists w_k \in M_l$, $\|w_k\|_m = 1$,

$$F_{1l}\left(w_k, a, \widetilde{b}_k\right) \leq 0. \tag{5.110}$$

Assume $w_k \rightharpoonup w^*$ in M_l. A trick mimicking the proof of Lemma 5.16 gets $w^* \neq 0$. Observe that for fixed $v \in N_l$,

$$\left|I\left(v + w_k, a, \widetilde{b}_k\right) - I\left(v + w_k, a, b\right)\right| \leq \frac{\left|\widetilde{b}_k - b\right|}{2}\left(\|v\|_{L^2}^2 + \frac{1}{\lambda_{l+1} + m}\right) \to 0. \tag{5.111}$$

Set $\varepsilon_k = \frac{\left|\widetilde{b}_k - b\right|}{2}\left(\|v\|_{L^2}^2 + \frac{1}{\lambda_{l+1}+m}\right)$, then we have

$$I\left(v + w_k, a, b\right) \leq \varepsilon_k \tag{5.112}$$

and an argument analogous to the proof of Lemma 5.16 yields

$$I\left(v + w^*, a, b\right) \leq 0. \tag{5.113}$$

Set $\|w^*\|_m = 1$, we obtain

$$M_l\left(a, b\right) \leq F_{1l}\left(w^*, a, b\right) \leq 0, \tag{5.114}$$

violating the hypothesis $M_l(a,b) > 0$. □

Proposition 5.20. *Under the hypotheses of Lemma 5.16, if* $(a,b) \in D_M^0 \setminus (\lambda_{l+1}, \lambda_{l+1})$*, then* $(a,b) \in \Sigma(A)$*.*

Proof. Argue by contradiction. Then there exists $\varepsilon_0 > 0$ small enough, $(a, b^*) \notin \Sigma(A)$ for $\forall b^* \in [b - \varepsilon_0, b + \varepsilon_0]$. Take two cases into account:

(i) $\Gamma(a,b) > \lambda_l$.

Employing Lemma 5.17, we get $\Gamma(a,b) < \lambda_{l+1} < \Lambda(a,b)$. By Lemma 5.18, for $\forall b_1 \in [b - \varepsilon_0, b)$, $\forall b_2 \in (b, b + \varepsilon_0]$, $M_l(a, b_1) > 0$, $M_l(a, b_2) < 0$. Invoking Lemma 5.11,

$$C_q\left(I_{(a,b-\varepsilon_0)}, 0\right) \cong \delta_{q d_l} G. \tag{5.115}$$

Similar to Theorem 1.1(iii) of [32], we obtain

$$C_{d_l}\left(I_{(a,b+\varepsilon_0)}, 0\right) \cong 0. \tag{5.116}$$

However, by Proposition 9.4,

$$C_q\left(I_{(a,b-\varepsilon_0)},0\right)\cong C_q\left(I_{(a,b+\varepsilon_0)},0\right), \tag{5.117}$$

contradicting the combination of (5.115) and (5.116).

(ii) $\Gamma(a,b)=\lambda_l$.

(1) $a=\lambda_l<b$.

Again by Lemma 5.18, $M_l(\lambda_l,b_1)>0$, $M_l(\lambda_l,b_2)<0$, for $b_1\in[b-\varepsilon_0,b)$, $b_2\in(b,b+\varepsilon_0]$. Arbitrarily choose $b_1^*\in[b-\varepsilon_0,b)$, $b_2^*\in(b,b+\varepsilon_0]$. An argument analogous to that of Lemma 5.19 indicates that $\exists\varepsilon^*>0$ sufficiently small, s.t., for $\forall\widetilde{a}\in(\lambda_l,\lambda_l+\varepsilon^*]$, $M_l\left(\widetilde{a},b_1^*\right)>0$. Notice that $\left(\widetilde{a},b_1^*\right)$, $\left(\widetilde{a},b_2^*\right)\notin\Sigma(A)$ and $M_l\left(\widetilde{a},b_2^*\right)<0$, a standard argument derives a contradiction

$$G\cong C_{d_l}\left(I_{(\lambda_l,b_1^*)},0\right)\cong C_{d_l}\left(I_{(\lambda_l,b_2^*)},0\right)\cong 0. \tag{5.118}$$

(2) $b=\lambda_l<a$.

Repeating the proof procedure of Lemma 5.18 shows that $\exists\widetilde{\varepsilon}>0$ suitably small, $M_l(\widetilde{a}_1,b)>0$, $M_l(\widetilde{a}_2,b)<0$, for $\forall\widetilde{a}_1\in[a-\widetilde{\varepsilon},a)$, $\forall\widetilde{a}_2\in(a,a+\widetilde{\varepsilon}]$. Fully imitating the trick furnished by (1), we conclude the proof. □

Lemma 5.21. *Under the hypotheses of Lemma 5.16, if $(a_i,b_i)\in D_M^0$, $a_1<a_2$, then $b_1>b_2$.*

Proof. If not, hence, by Lemma 5.16, $\exists w\in M_l$, $\|w\|_m=1$,

$$\begin{aligned}0&=M_l(a_1,b_1)=F_{1l}(w,a_1,b_1)\\&=\sup_{v\in N_l}I(v+w,a_1,b_1)\\&\geq\sup_{v\in N_l}I(v+w,a_2,b_2)=F_{1l}(w,a_2,b_2).\end{aligned} \tag{5.119}$$

Due to $M_l(a_2,b_2)=0$, we get

$$\sup_{v\in N_l}I(v+w,a_2,b_2)=0. \tag{5.120}$$

Thus, by Lemma 5.7, $\exists\theta(w,a_2,b_2)\in N_l$,

$$\begin{aligned}0&=I(\theta(w,a_2,b_2)+w,a_2,b_2)\\&\leq I(\theta(w,a_2,b_2)+w,a_1,b_1)\\&\leq I(\theta(w,a_1,b_1)+w,a_1,b_1)=0,\end{aligned} \tag{5.121}$$

and this yields

$$\left\|(\theta(w,a_2,b_2)+w)^-\right\|_{L^2}=0. \tag{5.122}$$

Observe that $w\neq 0\Rightarrow\theta(w,a_2,b_2)\neq 0$. On the other side,

$$\lambda_l \|\theta(w, a_2, b_2)\|_{L^2}^2 \geq \int_{\mathbb{R}^N} |\nabla\theta(w, a_2, b_2)|^2 + V(x)|\theta(w, a_2, b_2)|^2$$
$$= b_2 \|\theta(w, a_2, b_2)\|_{L^2}^2 \geq \lambda_l \|\theta(w, a_2, b_2)\|_{L^2}^2, \tag{5.123}$$

indicating that $\theta(w, a_2, b_2)$ is an eigenvector of $\lambda_l = b_2$. Set $\xi = \theta(w, a_2, b_2) + w$. As $\xi \geq 0$ a.e. on $\mathbb{R}^N$ and $\xi \perp \psi_1$, one derives $\xi = 0$. Out of the question! The conclusion follows. □

For fixed $a \in [\lambda_l, \sigma_0)$, if there exists $b \in D$, s.t., $(a, b) \in D_M^+ \cup D_M^0$, we denote

$$\widetilde{\nu}_l(a) = \sup\left\{b : (a, b) \in D_M^+ \cup D_M^0\right\}.$$

Lemma 5.22. *Under the hypothesis* (V_0)*, if* $(a, b) \in D$*, then* $b = \widetilde{\nu}_l(a) \Longleftrightarrow (a, b) \in D_M^0$.

Proof. "⇒" is determined by the combination of Lemma 5.19 and Lemma 5.24 below, and simultaneously Lemma 5.18 offers a powerful guarantee to "⇐". □

Denote

$$D^* = \{(a, b) \in D : \lambda_l \leq a \leq \lambda_{l+1} \leq b\};$$
$$D_* = \{(a, b) \in D : \lambda_l \leq b \leq \lambda_{l+1} \leq a\}.$$

Theorem 5.23. *Under the hypothesis* (V_0)*,* $\widetilde{\nu}_l(a)$ *is continuous on* $a \in \left[\lambda_l, \lambda_{l+1}\right]$ *as* $(a, \widetilde{\nu}_l(a)) \in D^*$*, and also on* $a \in \left[\lambda_{l+1}, \sigma_0\right)$ *as* $(a, \widetilde{\nu}_l(a)) \in D_*$.

Aiming at Theorem 5.23, we prove two lemmas below adapted to our needs:

Lemma 5.24. *Under the hypothesis* (V_0)*,* $M_l(a, b)$ *is continuous on* $(a, b) \in \widetilde{D}$.

Proof. Without loss of generality, assume that $(a_n, b_n) \in \widetilde{D}$, $(a_n, b_n) \to (a, b)$, and $a \leq b$. Divide the proof into three cases:

$\langle 1 \rangle$ $b \geq a > \lambda_l$.

We prove that for fixed $w \in M_l$, $\|w\|_m = 1$,

$$F_{1l}(w, a_n, b_n) \to F_{1l}(w, a, b). \tag{5.124}$$

Since

$$F_{1l}(w, a_n, b_n) \geq F_{1l}(w, \Lambda(a, b) + \varepsilon, \Lambda(a, b) + \varepsilon) > -\infty \tag{5.125}$$

for n large enough and $\varepsilon > 0$ suitably small, an argument analogous to that of Lemma 5.7 derives boundedness of $\|\theta(w, a_n, b_n)\|_m$. With the aid of boundedness of $\|\theta(w, a_n, b_n)\|_m$, we deduce from $\dim N_l < +\infty$ that there is a subsequence $\left\{\theta\left(w, a_{n_j}, b_{n_j}\right)\right\}_{j=1}^{+\infty}$,

$$\theta\left(w, a_{n_j}, b_{n_j}\right) \to v^* \text{ in } H_m^1 \cap N_l \tag{5.126}$$

as $j \to +\infty$, and then

$$I\left(\theta\left(w, a_{n_j}, b_{n_j}\right) + w, a_{n_j}, b_{n_j}\right) \to I\left(v^* + w, a, b\right). \tag{5.127}$$

Consequently,

$$\begin{aligned} & I\left(\theta\left(w,a,b\right)+w,a,b\right) \\ &= \lim_{j\to+\infty} I\left(\theta\left(w,a,b\right)+w,a_{n_j},b_{n_j}\right) \\ &\le \lim_{j\to+\infty} \sup_{v\in N_l} I\left(v+w,a_{n_j},b_{n_j}\right) \\ &= \lim_{j\to+\infty} I\left(\theta\left(w,a_{n_j},b_{n_j}\right)+w,a_{n_j},b_{n_j}\right) \\ &= I\left(v^*+w,a,b\right) \le I\left(\theta\left(w,a,b\right)+w,a,b\right), \end{aligned} \tag{5.128}$$

so we have

$$F_{1l}\left(w, a_{n_j}, b_{n_j}\right) \to F_{1l}\left(w, a, b\right), \tag{5.129}$$

alluding to (5.124). Hence,

$$\overline{\lim_{n\to+\infty}} M_l\left(a_n, b_n\right) \le \overline{\lim_{n\to+\infty}} F_{1l}\left(w, a_n, b_n\right) = F_{1l}\left(w, a, b\right), \tag{5.130}$$

and this yields

$$\overline{\lim_{n\to+\infty}} M_l\left(a_n, b_n\right) \le M_l\left(a, b\right). \tag{5.131}$$

On the other side, in view of definition, for $\varepsilon_n > 0$, $\varepsilon_n \to 0$, $\exists w_n \in M_l$, $\|w_n\|_m = 1$, s.t.

$$M_l\left(a_n, b_n\right) \ge F_{1l}\left(w_n, a_n, b_n\right) - \varepsilon_n. \tag{5.132}$$

We claim that

$$\lim_{n\to+\infty} \left|F_{1l}\left(w_n, a_n, b_n\right) - F_{1l}\left(w_n, a, b\right)\right| = 0. \tag{5.133}$$

Indeed, observe that

$$\begin{aligned} & \left\langle I_v'\left(\theta\left(w_n,a_n,b_n\right)+w_n,a_n,b_n\right) - I_v'\left(\theta\left(w_n,a,b\right)+w_n,a,b\right), \theta\left(w_n,a_n,b_n\right)-\theta\left(w_n,a,b\right)\right\rangle_m \\ &= \int_{\mathbb{R}^N} \left|\nabla\left(\theta\left(w_n,a_n,b_n\right)-\theta\left(w_n,a,b\right)\right)\right|^2 + V\left(\theta\left(w_n,a_n,b_n\right)-\theta\left(w_n,a,b\right)\right)^2 \\ &\quad -a\int_{\mathbb{R}^N} \left(\theta\left(w_n,a_n,b_n\right)-\theta\left(w_n,a,b\right)\right)^2 \end{aligned}$$

$$
\begin{aligned}
&- (b-a) \int\limits_{\mathbb{R}^N} \left[(\theta (w_n, a_n, b_n) + w_n)^+ - (\theta (w_n, a, b) + w_n)^+ \right] \\
&\cdot (\theta (w_n, a_n, b_n) - \theta (w_n, a, b)) \\
&- (a_n - a) \int\limits_{\mathbb{R}^N} (\theta (w_n, a_n, b_n) + w_n)^- \cdot (\theta (w_n, a_n, b_n) - \theta (w_n, a, b)) \\
&- (b_n - b) \int\limits_{\mathbb{R}^N} (\theta (w_n, a_n, b_n) + w_n)^+ \cdot (\theta (w_n, a_n, b_n) - \theta (w_n, a, b)) ,
\end{aligned}
\tag{5.134}
$$

and

$$
\begin{aligned}
&(\theta (w_n, a_n, b_n) + w_n)^+ - (\theta (w_n, a, b) + w_n)^+ \\
&\geq (\theta (w_n, a_n, b_n) - \theta (w_n, a, b))^- ,
\end{aligned}
\tag{5.135}
$$

and

$$
\begin{aligned}
&(\theta (w_n, a_n, b_n) + w_n)^+ - (\theta (w_n, a, b) + w_n)^+ \\
&\leq (\theta (w_n, a_n, b_n) - \theta (w_n, a, b))^+ .
\end{aligned}
\tag{5.136}
$$

Inserting (5.135) and (5.136) into (5.134), we get

$$
0 \leq \frac{\lambda_l - a}{\lambda_l + m} \| \theta (w_n, a_n, b_n) - \theta (w_n, a, b) \|_m^2 + \widetilde{\varepsilon}_n \tag{5.137}
$$

as $\widetilde{\varepsilon}_n \to 0$. Since $a > \lambda_l$, we derive

$$
\| \theta (w_n, a_n, b_n) - \theta (w_n, a, b) \|_m \to 0. \tag{5.138}
$$

The prediction (5.133) is confirmed accordingly. Therefore, (5.132) yields

$$
M_l (a_n, b_n) \geq F_{1l} (w_n, a, b) - \varepsilon_n^* \geq M_l (a, b) - \varepsilon_n^* \tag{5.139}
$$

as $\varepsilon_n^* \to 0$. The assertion is concluded by the combination of (5.131) and (5.139).

$\langle 2 \rangle$ $b > a = \lambda_l$.

A similar discussion gets (5.124) and (5.131). Based on the definition, for $\varepsilon_n > 0$, $\varepsilon_n \to 0$, $\exists w_n \in M_l$, $\|w_n\|_m = 1$, s.t.

$$
M_l (a_n, b_n) + \varepsilon_n \geq F_{1l} (w_n, a_n, b_n) , \tag{5.140}
$$

and then

$$
\underline{\lim}_{n \to +\infty} M_l (a_n, b_n) \geq \underline{\lim}_{n \to +\infty} (F_{1l} (w_n, a_n, b_n) - \varepsilon_n) . \tag{5.141}
$$

Notice that

$$\begin{aligned}
&\underline{\lim}_{n\to+\infty}\left(F_{1l}\left(w_n,a_n,b_n\right)-\varepsilon_n\right)\\
&\geq \underline{\lim}_{n\to+\infty}\left(I\left(\theta\left(w_n,\lambda_l,b\right)+w_n,a_n,b_n\right)-\varepsilon_n\right)\\
&\geq \underline{\lim}_{n\to+\infty} F_{1l}\left(w_n,\lambda_l,b\right)\\
&\quad+\underline{\lim}_{n\to+\infty}\left[I\left(\theta\left(w_n,\lambda_l,b\right)+w_n,a_n,b_n\right)-F_{1l}\left(w_n,\lambda_l,b\right)\right]\\
&=\underline{\lim}_{n\to+\infty} F_{1l}\left(w_n,\lambda_l,b\right)\geq M_l\left(\lambda_l,b\right).
\end{aligned} \tag{5.142}$$

Combining (5.131), (5.141) and (5.142), we obtain

$$M_l\left(\lambda_l,b\right)\geq \overline{\lim}_{n\to+\infty} M_l\left(a_n,b_n\right)\geq \underline{\lim}_{n\to+\infty} M_l\left(a_n,b_n\right)\geq M_l\left(\lambda_l,b\right), \tag{5.143}$$

alluding to $M_l\left(a_n,b_n\right)\to M_l\left(\lambda_l,b\right)$.

$\langle 3\rangle$ $a=b=\lambda_l$.

Observe that $M_l\left(a_n,b_n\right)\leq M_l\left(\lambda_l,\lambda_l\right)$ shows (5.131). In virtue of the definition, for $\varepsilon_n>0$, $\varepsilon_n\to 0$, $\exists w_n\in M_l$, $\|w_n\|_m=1$, (5.141) follows. As $\theta\left(w_n,\lambda_l,\lambda_l\right)=0$, again by (5.131), (5.141) and (5.142), we reach the conclusion as desired. The proof is complete. □

Lemma 5.25. *Under the hypotheses of Lemma 5.16, if $(a,b)\in D_M^0$ and $\Gamma(a,b)>\lambda_l$, then $\exists\delta>0$, $\forall a^*\in[a-\delta,a+\delta]$, $\exists b^*\in\mathbb{R}$, s.t., $(a^*,b^*)\in D_M^0$. If $(a,b)\in D_M^0$ and $a=\lambda_l<b$, then $\exists\widetilde{\delta}>0$, $\forall a^*\in\left[a,a+\widetilde{\delta}\right]$, $\exists b^*\in\mathbb{R}$, s.t., $(a^*,b^*)\in D_M^0$. If $(a,b)\in D_M^0$ and $b=\lambda_l<a$, then $\exists\overline{\delta}>0$, $\forall a^*\in\left[a-\overline{\delta},a\right]$, $\exists b^*\in\mathbb{R}$, s.t., $(a^*,b^*)\in D_M^0$.*

Proof. For the case $\Gamma(a,b)>\lambda_l$, by Lemma 5.18, $\forall b_1\in[\lambda_l,b)$, $\forall b_2\in\left(b,\widetilde{\theta}_{l+1}\right)$, $(a,b_1)\in D_M^+$, $(a,b_2)\in D_M^-$. Take $\varepsilon>0$ sufficiently small, $\forall b_1\in[b-\varepsilon,b)$, $\forall b_2\in(b,b+\varepsilon]$, $(a,b_1)\in D_M^+$, $(a,b_2)\in D_M^-$. Using Lemma 5.24, $\exists\delta_1>0$, $\forall a^*\in[a-\delta_1,a+\delta_1]$, $(a^*,b+\varepsilon)\in D_M^-$, and also $\exists\delta_2>0$, $\forall a^*\in[a-\delta_2,a+\delta_2]$, $(a^*,b-\varepsilon)\in D_M^+$. Taking $\delta=\Gamma(\delta_1,\delta_2)$ concludes the proof. A similar argument tackles with the case $\Gamma(a,b)=\lambda_l<\Lambda(a,b)$. □

Proof of Theorem 5.23. In view of Lemma 5.25, suppose that $(a_n,\widetilde{\nu}_l(a_n))$, $(a,\widetilde{\nu}_l(a))\in D_M^0$ and $a_n\to a$, we prove $\widetilde{\nu}_l(a_n)\to\widetilde{\nu}_l(a)$. Without loss of generality, assume $(a,\widetilde{\nu}_l(a))\in D^*$. Take two cases into account:

(1) $\{a_n\}$ is strictly increasing.

Using Lemma 5.21 we obtain that $\{\widetilde{\nu}_l(a_n)\}_{n=1}^{+\infty}$ is strictly decreasing and thus convergent. Assume $\widetilde{\nu}_l(a_n)\to b^*$. As $\widetilde{\nu}_l(a_n)>\widetilde{\nu}_l(a)$ for fixed $n\in\mathbb{N}$, we get

$$\sigma_0>\widetilde{\nu}_l(a_n)>b^*\geq\widetilde{\nu}_l(a)\geq\lambda_{l+1}\geq a>a_n\geq\lambda_l, \tag{5.144}$$

alluding to (a,b^*), $(a_n,b^*)\in D^*$. Hence, by Lemma 5.22,

$$\begin{aligned}
0&=M_l\left(a_n,\widetilde{\nu}_l(a_n)\right)\\
&=\inf_{w\in M_l,\|w\|_m=1} F_{1l}\left(w,a_n,\widetilde{\nu}_l(a_n)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{w\in M_l,\|w\|_m=1}\sup_{v\in N_l} I\left(v+w,a_n,\widetilde{\nu}_l\left(a_n\right)\right)\\
&\le \inf_{w\in M_l,\|w\|_m=1}\sup_{v\in N_l} I\left(v+w,a_n,b^*\right)\\
&= M_l\left(a_n,b^*\right).
\end{aligned}
\tag{5.145}
$$

By Lemma 5.24, we obtain $M_l\left(a,b^*\right)\ge 0$ so $b^*=\widetilde{\nu}_l\left(a\right)$.

(2) $\{a_n\}$ is strictly decreasing.

Similarly, we yield that $\{\widetilde{\nu}_l\left(a_n\right)\}_{n=1}^{+\infty}$ is strictly increasing and thus convergent. Assume $\widetilde{\nu}_l\left(a_n\right)\to b_0^*$. Since $\widetilde{\nu}_l\left(a_n\right)<\widetilde{\nu}_l\left(a\right)$ for fixed $n\in\mathbb{N}$, we get

$$
\lambda_l\le\widetilde{\nu}_l\left(a_n\right)<b_0^*\le\widetilde{\nu}_l\left(a\right),
\tag{5.146}
$$

showing $\left(a,b_0^*\right)$, $\left(a_n,b_0^*\right)\in D$. Again by Lemma 5.22,

$$
\begin{aligned}
0&=M_l\left(a_n,\widetilde{\nu}_l\left(a_n\right)\right)\\
&= \inf_{w\in M_l,\|w\|_m=1} F_{1l}\left(w,a_n,\widetilde{\nu}_l\left(a_n\right)\right)\\
&= \inf_{w\in M_l,\|w\|_m=1}\sup_{v\in N_l} I\left(v+w,a_n,\widetilde{\nu}_l\left(a_n\right)\right)\\
&\ge \inf_{w\in M_l,\|w\|_m=1}\sup_{v\in N_l} I\left(v+w,a_n,b_0^*\right)\\
&= M_l\left(a_n,b_0^*\right).
\end{aligned}
\tag{5.147}
$$

Resorting to Lemma 5.24, $M_l\left(a,b_0^*\right)\le 0$ and then $b_0^*=\widetilde{\nu}_l\left(a\right)$, ending the proof. □

Next we turn our eyesight to analysis of the maximal curves.

Lemma 5.26. *Under the hypothesis* (V_0)*, if*
(i) $\Lambda\left(a,b\right)<\widetilde{\theta}_{l+1}$*;*
or
(ii) $\Gamma\left(a,b\right)<\Lambda\left(a,b\right)=\widetilde{\theta}_{l+1}<\sigma_0$*,*
then $\exists v_0\in N_{l+1}$*,* $\|v_0\|_m=1$*, s.t.*

$$
F_{2l+1}\left(v_0,a,b\right)=m_{l+1}\left(a,b\right).
\tag{5.148}
$$

Proof. Set $c=m_{l+1}\left(a,b\right)$. Notice that for fixed $v\in N_{l+1}$, $\|v\|_m=1$,

$$
\begin{aligned}
F_{2l+1}\left(v,a,b\right)&\le I\left(v,a,b\right)\\
&\le\frac{\lambda_{l+1}-\Gamma\left(a,b\right)}{2\left(\lambda_{l+1}+m\right)},
\end{aligned}
\tag{5.149}
$$

so we get $c<+\infty$.

On the other side, as $\Lambda\left(a,b\right)\le\widetilde{\theta}_{l+1}$, for $\forall w\in M_{l+1}$,

$$I(v+w,a,b) \geq \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 + Vv^2 - \frac{\Lambda(a,b)}{2}\|v\|_{L^2}^2$$

$$\geq \frac{\lambda_1 - \Lambda(a,b)}{2(\lambda_1+m)} \tag{5.150}$$

and this shows $c > -\infty$.

In virtue of the definition, there exists a sequence $\{v_k\}_{k=1}^{+\infty} \subset N_{l+1}$, $\|v_k\|_m = 1$, s.t.

$$F_{2l+1}(v_k,a,b) \to c. \tag{5.151}$$

Due to $\dim N_{l+1} < +\infty$, assume $v_k \to v_0$. Observe that for fixed $w \in M_{l+1}$, $\forall \varepsilon > 0$,

$$\lim_{k\to+\infty} F_{2l+1}(v_k,a,b) \leq \lim_{k\to+\infty} I(v_k+w,a,b) = I(v_0+w,a,b), \tag{5.152}$$

implying that $c \leq F_{2l+1}(v_0,a,b)$. The assertion follows. □

Lemma 5.27. *Under the hypotheses of Lemma 5.26, if $(a,b) \in B_m^0 \setminus (\lambda_{l+1},\lambda_{l+1})$, then $(a,b) \in B^*$ or $(a,b) \in B_*$.*

Proof. For fixed $(a_0,b_0) \in Q_{l+1}^* \setminus (B^* \cup B_*)$, take account of the following cases:

(1) $\lambda_l \leq \Gamma(a_0,b_0) \leq \Lambda(a_0,b_0) \leq \lambda_{l+1}$.

(i) $\Lambda(a_0,b_0) < \lambda_{l+1}$.

For fixed $\varphi_0 \in E(\lambda_{l+1})$, $\|\varphi_0\|_m = 1$, and $\forall w \in M_{l+1}$,

$$\begin{aligned} I(\varphi_0+w,a_0,b_0) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\varphi_0|^2 + V(x)\varphi_0^2 + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 + V(x)w^2 \\ &\quad - \frac{a_0}{2}\int_{\mathbb{R}^N} \left|(\varphi_0+w)^-\right|^2 - \frac{b_0}{2}\int_{\mathbb{R}^N} \left|(\varphi_0+w)^+\right|^2 \\ &\geq \frac{1}{2}\left(1 - \frac{\Lambda(a_0,b_0)+m}{\lambda_{l+1}+m}\right) + \frac{1}{2}\left[\widetilde{\theta}_{l+1} - \Lambda(a_0,b_0)\right]\cdot\|w\|_{L^2}^2 \\ &\geq \frac{\lambda_{l+1} - \Lambda(a_0,b_0)}{2(\lambda_{l+1}+m)} \end{aligned} \tag{5.153}$$

and thus

$$m_{l+1}(a_0,b_0) \geq F_{2l+1}(\varphi_0,a_0,b_0) \geq \frac{\lambda_{l+1} - \Lambda(a_0,b_0)}{2(\lambda_{l+1}+m)} > 0. \tag{5.154}$$

(ii) $\Lambda(a_0,b_0) = \lambda_{l+1}$.

Clearly, $m_{l+1}(a_0,b_0) \geq 0$. If $m_{l+1}(a_0,b_0) = 0$, then for fixed $\varphi_0 \in E(\lambda_{l+1})$, $\|\varphi_0\|_m = 1$, s.t.

$$\begin{aligned}0 &\geq F_{2l+1}(\varphi_0, a_0, b_0) = I(\varphi_0 + \tau(\varphi_0, a_0, b_0), a_0, b_0)\\ &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\varphi_0|^2 + V(x)|\varphi_0|^2 + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\tau(\varphi_0, a_0, b_0)|^2 + V(x)|\tau(\varphi_0, a_0, b_0)|^2\\ &\quad - \frac{a_0}{2}\int_{\mathbb{R}^N} \left|(\varphi_0 + \tau(\varphi_0, a_0, b_0))^-\right|^2 - \frac{b_0}{2}\int_{\mathbb{R}^N} \left|(\varphi_0 + \tau(\varphi_0, a_0, b_0))^+\right|^2.\end{aligned} \tag{5.155}$$

We claim that

$$\begin{aligned}&a_0\int_{\mathbb{R}^N} \left|(\varphi_0 + \tau(\varphi_0, a_0, b_0))^-\right|^2 + b_0\int_{\mathbb{R}^N} \left|(\varphi_0 + \tau(\varphi_0, a_0, b_0))^+\right|^2\\ &< \lambda_{l+1}\left(\|\varphi_0\|_{L^2}^2 + \|\tau(\varphi_0, a_0, b_0)\|_{L^2}^2\right).\end{aligned} \tag{5.156}$$

Without loss of generality, assume $b_0 = \lambda_{l+1} > a_0$. If (5.156) is violated, we get

$$\left\|(\varphi_0 + \tau(\varphi_0, a_0, b_0))^-\right\|_{L^2} = 0, \tag{5.157}$$

and then $\varphi_0 + \tau(\varphi_0, a_0, b_0) \geq 0$ a.e. on $\mathbb{R}^N$.

As $\left\langle I'_w(\varphi_0 + \tau(\varphi_0, a_0, b_0), a_0, b_0), \widetilde{w}\right\rangle = 0$ for $\forall \widetilde{w} \in M_{l+1}$, we have

$$\int_{\mathbb{R}^N} \nabla\tau(\varphi_0, a_0, b_0)\nabla\widetilde{w} + V(x)\tau(\varphi_0, a_0, b_0)\widetilde{w} = \lambda_{l+1}\langle\tau(\varphi_0, a_0, b_0), \widetilde{w}\rangle_{L^2}. \tag{5.158}$$

Take $\widetilde{w} = \tau(\varphi_0, a_0, b_0)$, (5.158) yields

$$\begin{aligned}&\widetilde{\theta}_{l+1}\|\tau(\varphi_0, a_0, b_0)\|_{L^2}^2\\ &\leq \int_{\mathbb{R}^N} |\nabla\tau(\varphi_0, a_0, b_0)|^2 + V(x)|\tau(\varphi_0, a_0, b_0)|^2\\ &= \lambda_{l+1}\|\tau(\varphi_0, a_0, b_0)\|_{L^2}^2\end{aligned} \tag{5.159}$$

and this shows $\varphi_0 \geq 0$ a.e. on $\mathbb{R}^N$. As $\varphi_0 \perp \psi_1$, it follows that $\varphi_0 = 0$ a.e. on $\mathbb{R}^N$, contradicting $\|\varphi_0\|_m = 1$. The claim is proved.

Combining (5.155) and (5.156), we derive a self-contradictory inequality

$$\begin{aligned}0 &\geq I(\varphi_0 + \tau(\varphi_0, a_0, b_0), a_0, b_0)\\ &> \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\tau(\varphi_0, a_0, b_0)|^2 + V(x)|\tau(\varphi_0, a_0, b_0)|^2 - \frac{\lambda_{l+1}}{2}\|\tau(\varphi_0, a_0, b_0)\|_{L^2}^2\\ &\geq \frac{\widetilde{\theta}_{l+1} - \lambda_{l+1}}{2}\|\tau(\varphi_0, a_0, b_0)\|_{L^2}^2 \geq 0,\end{aligned} \tag{5.160}$$

also alluding to $m_{l+1}(a_0, b_0) > 0$.

(2) $\lambda_{l+1} \le \Gamma(a_0, b_0) \le \Lambda(a_0, b_0) \le \widetilde{\theta}_{l+1}$.

If $\Gamma(a_0, b_0) > \lambda_{l+1}$, for fixed $v \in N_{l+1}$, $\|v\|_m = 1$,

$$I(v, a_0, b_0) = \frac{1}{2} \int\limits_{\mathbb{R}^N} |\nabla v|^2 + V(x) v^2 - a_0 \left(v^-\right)^2 - b_0 \left(v^+\right)^2 \le -\frac{\Gamma(a_0, b_0) - \lambda_{l+1}}{2(\lambda_{l+1} + m)} < 0, \tag{5.161}$$

and thus $m_{l+1}(a_0, b_0) < 0$. If $\Gamma(a_0, b_0) = \lambda_{l+1}$, without loss of generality, assume $\widetilde{\theta}_{l+1} \ge b_0 > a_0 = \lambda_{l+1}$. Evidently, $I(v, a_0, b_0) < 0$ unless

$$a_0 \left\|v^-\right\|_{L^2}^2 + b_0 \left\|v^+\right\|_{L^2}^2 = \lambda_{l+1} \|v\|_{L^2}^2, \tag{5.162}$$

so we are merely to deal with this case.

Observe that (5.162) $\Rightarrow v \le 0$ a.e. on $\mathbb{R}^N$. Hence, if $I(v, a_0, b_0) = 0$, we have

$$\langle Av, v\rangle_{L^2} = \lambda_{l+1} \|v\|_{L^2}, \tag{5.163}$$

indicating that v is an eigenvector of λ_{l+1}, and consequently $v \perp \psi_1 \Rightarrow v = 0$, contradicting $\|v\|_m = 1$. Therefore, $I(v, a_0, b_0) < 0$, and by Lemma 5.26, we also derive $m_{l+1}(a_0, b_0) < 0$. The proof is complete. □

Lemma 5.28. *Under the hypotheses of Lemma 5.26, if $(a, b) \in F_m^0$, $\lambda_1 < \Gamma(a, b) \le \Lambda(a, b) < \widetilde{\theta}_{l+1}$ or $\lambda_1 < b < \lambda_{l+1} < a = \widetilde{\theta}_{l+1} < \sigma_0$, then for $\forall b_1 \in (-\infty, b)$, $\forall b_2 \in \left(b, \widetilde{\theta}_{l+1}\right]$, $(a, b_1) \in F_m^+$, $(a, b_2) \in F_m^-$. In addition, if $(a, b) \in F_m^0$, $\lambda_1 < a < \lambda_{l+1} < b = \widetilde{\theta}_{l+1} < \sigma_0$, then for $\forall \widetilde{b} \in (-\infty, b)$, $\left(a, \widetilde{b}\right) \in F_m^+$.*

Aiming at Lemma 5.28, we prove the following lemma adapted to our needs:

Lemma 5.29. *Under the hypotheses of Lemma 5.26, if $(a, b) \in F_m^0$, $\lambda_1 < \Gamma(a, b) \le \Lambda(a, b) \le \widetilde{\theta}_{l+1} < \sigma_0$, then $(a, b) \in \Sigma(A)$.*

Proof. We divide the proof into two cases:

(1) $a \le b$. We deduce from Lemma 5.27 that for fixed $(a_0, b_0) \in \mathbb{R}^2 \setminus (\lambda_{l+1}, \lambda_{l+1})$, if $\Lambda(a_0, b_0) \le \lambda_{l+1}$, $m_{l+1}(a_0, b_0) > 0$, and if $\Gamma(a_0, b_0) \ge \lambda_{l+1}$, $m_{l+1}(a_0, b_0) < 0$. Therefore, it follows that $a < \lambda_{l+1} < b$.

Suppose by contradiction, $(a, b) \notin \Sigma(A)$, then there exists $\varepsilon_0 > 0$ sufficiently small, $(a^*, b) \notin \Sigma(A)$ for $\forall a^* \in [a - \varepsilon_0, a + \varepsilon_0]$. We claim that $\forall a_1^* \in [a - \varepsilon_0, a)$, $m_{l+1}\left(a_1^*, b\right) > 0$. Otherwise, $\exists a_1^* \in [a - \varepsilon_0, a)$, $m_{l+1}\left(a_1^*, b\right) = 0$. By Lemma 5.26, $\exists v \in N_{l+1}$, $\|v\|_m = 1$,

$$F_{2l+1}(v, a, b) = m_{l+1}(a, b) = 0. \tag{5.164}$$

We first show

$$I\left(v + \tau\left(v, a_1^*, b\right), a_1^*, b\right) > I\left(v + \tau\left(v, a_1^*, b\right), a, b\right). \tag{5.165}$$

By way of negation, we obtain $v + \tau\left(v, a_1^*, b\right) \ge 0$.

Notice that $\left\langle I_w'\left(v+\tau\left(v,a_1^*,b\right),a_1^*,b\right),\widetilde{w}\right\rangle=0$ for $\forall\widetilde{w}\in M_{l+1}$, i.e.,

$$\int\limits_{\mathbb{R}^N}\nabla\tau\left(v,a_1^*,b\right)\nabla\widetilde{w}+V(x)\,\tau\left(v,a_1^*,b\right)\widetilde{w}=b\left\langle\tau\left(v,a_1^*,b\right),\widetilde{w}\right\rangle_{L^2},\tag{5.166}$$

yielding

$$\begin{aligned}&\widetilde{\theta}_{l+1}\left\|\tau\left(v,a_1^*,b\right)\right\|_{L^2}^2\\&\le\int\limits_{\mathbb{R}^N}\left|\nabla\tau\left(v,a_1^*,b\right)\right|^2+V(x)\left|\tau\left(v,a_1^*,b\right)\right|^2\\&=b\left\|\tau\left(v,a_1^*,b\right)\right\|_{L^2}^2.\end{aligned}\tag{5.167}$$

If $b=\widetilde{\theta}_{l+1}<\sigma_0$, notice that

$$\begin{aligned}0&=m_{l+1}\left(a_1^*,b\right)\\&\ge F_{2l+1}\left(v,a_1^*,b\right)\\&=I\left(v+\tau\left(v,a_1^*,b\right),a,b\right)\\&\ge F_{2l+1}(v,a,b)\\&=m_{l+1}(a,b)=0,\end{aligned}\tag{5.168}$$

and we immediately derive a self-contradictory inequality

$$\begin{aligned}0&=I\left(v+\tau\left(v,a_1^*,b\right),a,b\right)\\&=\frac{1}{2}\int\limits_{\mathbb{R}^N}|\nabla v|^2+V(x)\,v^2-\frac{b}{2}\|v\|_{L^2}^2<0,\end{aligned}\tag{5.169}$$

alluding to (5.165). Clearly $\tau\left(v,a_1^*,b\right)=0$ if $b<\widetilde{\theta}_{l+1}$, (5.165) also follows from (5.171).

(5.165) indicates that $m_{l+1}\left(a_1^*,b\right)>0$, violating the hypothesis $m_{l+1}\left(a_1^*,b\right)=0$. The claim is thus proved.

Along the same lines, $\forall a_2^*\in(a,a+\varepsilon_0]$, $m_{l+1}\left(a_2^*,b\right)<0$. Hence, an argument resembling Theorem 1.1(iv) and (v) of [32], gives rise to

$$C_{d_{l+1}}\left(I_{\left(a_1^*,b\right)},0\right)\cong 0,\ C_q\left(I_{\left(a_2^*,b\right)},0\right)\cong\delta_{qd_{l+1}}G.\tag{5.170}$$

However, by Proposition 9.4,

$$C_q\left(I_{\left(a_1^*,b\right)},0\right)\cong C_q\left(I_{\left(a_2^*,b\right)},0\right),\tag{5.171}$$

contradicting (5.170). The assertion follows.

(2) $b\le a$.

Employing Lemma 5.27 gets $b < \lambda_{l+1} < a$. Suppose to the contrary that $(a, b) \notin \Sigma(A)$. Then there exists $\varepsilon > 0$ suitably small, $(a, b^*) \notin \Sigma(A)$ for $\forall b^* \in [b-\varepsilon, b+\varepsilon]$. A standard argument concludes the proof. □

Proof of Lemma 5.28. Obviously, for $\forall \widetilde{b} < b$, $m_{l+1}\left(a, \widetilde{b}\right) \geq m_{l+1}(a, b)$. By way of negation, $\exists \widetilde{b} \in (-\infty, b)$, $m_{l+1}\left(a, \widetilde{b}\right) = 0$. Using Lemma 5.26, $\exists v \in N_{l+1}$, $\|v\|_m = 1$, s.t.

$$F_{2l+1}(v, a, b) = m_{l+1}(a, b) = 0. \tag{5.172}$$

We claim

$$I\left(v + \tau\left(v, a, \widetilde{b}\right), a, \widetilde{b}\right) > I\left(v + \tau\left(v, a, \widetilde{b}\right), a, b\right). \tag{5.173}$$

Argue by contradiction. If $\Lambda(a, b) < \widetilde{\theta}_{l+1}$, a standard argument shows $v \leq 0$ and $\tau\left(v, a, \widetilde{b}\right) = \tau(v, a, b) = 0$, and then by Lemma 5.4 and Lemma 5.29, we infer that a is an eigenvalue of A and so (5.173) follows immediately as $a \notin \sigma_{\text{dis}}(A)$. If $a \in \sigma_{\text{dis}}(A)$, it follows that v is an eigenvector of a and accordingly the combination of $v \perp \psi_1$ and $v \leq 0$ determines $v = 0$, conflicting with $\|v\|_m = 1$. In like manner, we get $(a, b_2) \in F_m^-$ for $\forall b_2 \in \left(b, \widetilde{\theta}_{l+1}\right]$.

Quite similarly, we can treat the case $\Gamma(a, b) < \lambda_{l+1} < \Lambda(a, b) = \widetilde{\theta}_{l+1} < \sigma_0$, ending the proof of Lemma 5.28. □

Denote

$$F^* = \left\{(a, b) \in \mathbb{R}^2 : -\infty < a \leq \lambda_{l+1} \leq b \leq \widetilde{\theta}_{l+1}\right\};$$
$$F_* = \left\{(a, b) \in \mathbb{R}^2 : -\infty < b \leq \lambda_{l+1} \leq a \leq \widetilde{\theta}_{l+1}\right\}.$$

Lemma 5.30. *Under the hypotheses of Lemma 5.26, if $(a_i, b_i) \in F_m^0$, $\lambda_1 < \Gamma(a_i, b_i) \leq \Lambda(a_i, b_i) < \widetilde{\theta}_{l+1}$ or $\lambda_1 < \Gamma(a_i, b_i) < \lambda_{l+1} < \Lambda(a_i, b_i) \leq \widetilde{\theta}_{l+1} < \sigma_0$, $i = 1, 2$, and $a_1 < a_2$, then $b_1 > b_2$.*

Proof. In terms of Lemma 5.27, if $(a, b) \neq (\lambda_{l+1}, \lambda_{l+1})$, then $m_{l+1}(a, b) > 0$ as $\lambda_l \leq \Gamma(a, b) \leq \Lambda(a, b) \leq \lambda_{l+1}$, and $m_{l+1}(a, b) < 0$ as $\lambda_{l+1} \leq \Gamma(a, b) \leq \Lambda(a, b) \leq \widetilde{\theta}_{l+1}$, so we are merely to deal with two cases: $\{(a_1, b_1), (a_2, b_2)\} \subset F^*$ or $\{(a_1, b_1), (a_2, b_2)\} \subset F_*$. Suppose to the contrary that $b_1 \leq b_2$. Actually, the contradiction originates directly from the combination of Lemma 5.26 and the proof of Lemma 5.29. □

Lemma 5.31. *Under the hypothesis (V_0), if $(a, b) \in F$, $m_{l+1}(a, b) < 0$, then there exists $\varepsilon_0 > 0$, $\forall \widetilde{b} \in [b - \varepsilon_0, b)$, $m_{l+1}\left(a, \widetilde{b}\right) < 0$.*

Proof. Otherwise, $\exists \widetilde{b}_k \to b$, $\widetilde{b}_k < b$, s.t. $m_{l+1}\left(a, \widetilde{b}_k\right) \geq 0$. Hence, for $\varepsilon_k > 0$, $\varepsilon_k \to 0$, $\exists v_k \in N_{l+1}$, $\|v_k\|_m = 1$, for fixed $w \in M_{l+1}$,

$$I\left(v_k + w, a, \widetilde{b}_k\right) \geq F_{2l+1}\left(v_k, a, \widetilde{b}_k\right) \geq m_{l+1}\left(a, \widetilde{b}_k\right) - \varepsilon_k \geq -\varepsilon_k. \tag{5.174}$$

Assume $v_k \to v_0$ and let $k \to +\infty$,

$$I\left(v_0+w,a,b\right)\geq 0 \tag{5.175}$$

and then yields $m_{l+1}(a,b)\geq 0$, violating the hypothesis $m_{l+1}(a,b)<0$. □

Define

$$\widetilde{\mu}_{l+1}(a)=\inf\left\{b:(a,b)\in F_m^0\cup F_m^-\right\}.$$

Lemma 5.32. *Under the hypotheses of Lemma 5.26, if $(a,b)\in F$, $\lambda_1<\Gamma(a,b)\leq\Lambda(a,b)<\widetilde{\theta}_{l+1}$ or $\lambda_1<\Gamma(a,b)<\lambda_{l+1}<\Lambda(a,b)=\widetilde{\theta}_{l+1}<\sigma_0$, then $b=\widetilde{\mu}_{l+1}(a)\Longleftrightarrow(a,b)\in F_m^0$.*

Proof. "$\Rightarrow$" originates from the combination of Lemma 5.31 and Lemma 5.34 below while "$\Leftarrow$" is concluded by Lemma 5.28. □

Theorem 5.33. *Under the hypotheses of Lemma 5.26, if $\widetilde{\theta}_{l+1}<\sigma_0$, then $\widetilde{\mu}_{l+1}(a)$ is continuous on $(\lambda_1,\lambda_{l+1}]$ as $(a,\widetilde{\mu}_{l+1}(a))\in F^*$ and also on $[\lambda_{l+1},\widetilde{\theta}_{l+1}]$ as $(a,\widetilde{\mu}_{l+1}(a))\in F_*$. In addition, if $\widetilde{\theta}_{l+1}=\sigma_0$, then $\widetilde{\mu}_{l+1}(a)$ is continuous on $(\lambda_1,\lambda_{l+1}]$ as $(a,\widetilde{\mu}_{l+1}(a))\in(F^*)^\circ$ and also on $[\lambda_{l+1},\widetilde{\theta}_{l+1})$ as $(a,\widetilde{\mu}_{l+1}(a))\in(F_*)^\circ$, where $(F^*)^\circ$ and $(F_*)^\circ$ denote the interior of F^* and F_* respectively.*

Two lemmas below serve as indispensable supplementary means to verify Theorem 5.33.

Lemma 5.34. *Under the hypotheses of Lemma 5.26, $m_{l+1}(a,b)$ is continuous on $(a,b)\in F$ if $\widetilde{\theta}_{l+1}<\sigma_0$, and also on $(a,b)\in F^\circ\cup\left\{\left(\widetilde{\theta}_{l+1},\widetilde{\theta}_{l+1}\right)\right\}$ if $\widetilde{\theta}_{l+1}=\sigma_0$.*

Proof. Without loss of generality, suppose that $(a_n,b_n)\in F$, $(a_n,b_n)\to(a,b)$, $a\leq b$. Take two cases into account:

⟨1⟩ $a\leq b<\widetilde{\theta}_{l+1}$ or $a<b=\widetilde{\theta}_{l+1}<\sigma_0$.

We claim that for fixed $v\in N_{l+1}$, $\|v\|_m=1$,

$$F_{2l+1}(v,a_n,b_n)\to F_{2l+1}(v,a,b). \tag{5.176}$$

Take $\varepsilon>0$. As

$$F_{2l+1}(v,a_n,b_n)\leq F_{2l+1}(v,a-\varepsilon,a-\varepsilon)\leq I(v,a-\varepsilon,a-\varepsilon)<+\infty, \tag{5.177}$$

an argument analogous to that of Lemma 5.1 derives the boundedness of $\|\tau(v,a_n,b_n)\|_m$. Assume that a renamed subsequence (a_n,b_n), s.t. $\tau(v,a_n,b_n)\rightharpoonup w^*$ in H_m^1, and thus

$$\underline{\lim}_{j\to+\infty} I\left(v+\tau(v,a_n,b_n),a,b\right)\geq I\left(v+w^*,a,b\right). \tag{5.178}$$

Hence, for $\forall w\in M_{l+1}$,

$$\begin{aligned} & I\left(v+\tau(v,a,b),a,b\right)\\ \leq\, & I\left(v+w^*,a,b\right)\end{aligned}$$

$$
\begin{aligned}
&\le \varliminf_{j\to+\infty} I\left(v+\tau\left(v,a_n,b_n\right),a,b\right) \\
&\le \varlimsup_{j\to+\infty} I\left(v+\tau\left(v,a_n,b_n\right),a,b\right) \\
&\le \varlimsup_{j\to+\infty} \left[I\left(v+\tau\left(v,a_n,b_n\right),a,b\right)-I\left(v+\tau\left(v,a_n,b_n\right),a_n,b_n\right)\right] \\
&\quad+\varlimsup_{j\to+\infty} I\left(v+\tau\left(v,a_n,b_n\right),a_n,b_n\right) \\
&\le \varlimsup_{j\to+\infty} I\left(v+w,a_n,b_n\right) \\
&= I\left(v+w,a,b\right),
\end{aligned}
\tag{5.179}
$$

yielding $w^* = \tau(v,a,b)$, and accordingly we get (5.176). Therefore,

$$
\varliminf_{n\to+\infty} m_{l+1}\left(a_n,b_n\right) \ge \varliminf_{n\to+\infty} F_{2l+1}\left(v,a_n,b_n\right) = F_{2l+1}\left(v,a,b\right), \tag{5.180}
$$

indicating that

$$
\varliminf_{n\to+\infty} m_{l+1}\left(a_n,b_n\right) \ge m_{l+1}\left(a,b\right). \tag{5.181}
$$

The other side of the shield, for $\varepsilon_n > 0$, $\varepsilon_n \to 0$, $\exists v_n \in N_{l+1}$, $\|v_n\|_m = 1$, s.t.

$$
m_{l+1}\left(a_n,b_n\right) \le F_{2l+1}\left(v_n,a_n,b_n\right) + \varepsilon_n. \tag{5.182}
$$

Assume $v_n \to v_0$. For fixed $w \in M_{l+1}$,

$$
\varlimsup_{n\to+\infty} F_{2l+1}\left(v_n,a_n,b_n\right) \le \varlimsup_{n\to+\infty} I\left(v_n+w,a_n,b_n\right) = I\left(v_0+w,a,b\right) \tag{5.183}
$$

and consequently,

$$
\varlimsup_{n\to+\infty} F_{2l+1}\left(v_n,a_n,b_n\right) \le F_{2l+1}\left(v_0,a,b\right) \le m_{l+1}\left(a,b\right). \tag{5.184}
$$

Combining (5.181), (5.182) and (5.184) we arrive at the conclusion.
$\langle 2\rangle$ $a = b = \widetilde{\gamma} \le \sigma_0$.
Thanks to $m_{l+1}\left(a_n,b_n\right) \ge m_{l+1}\left(a,b\right)$, the assertion follows. □

Lemma 5.35. *Under the hypotheses of Lemma 5.26, if $(a,b) \in F_m^0$, $\lambda_1 < \Gamma(a,b) \le \Lambda(a,b) < \widetilde{\theta}_{l+1}$, then $\exists \delta_1 > 0$, $\forall \widetilde{a} \in [a-\delta_1, a+\delta_1]$, $\exists \widetilde{b} \in \mathbb{R}$, s.t., $\left(\widetilde{a},\widetilde{b}\right) \in F_m^0$, and if $(a,b) \in F_m^0$, $\lambda_1 < a < \lambda_{l+1} < b = \widetilde{\theta}_{l+1} < \sigma_0$, then $\exists \delta_2 > 0$, $\forall \widetilde{a} \in [a, a+\delta_2]$, $\exists \widetilde{b} \in \mathbb{R}$, s.t., $\left(\widetilde{a},\widetilde{b}\right) \in F_m^0$, and if $(a,b) \in F_m^0$, $\lambda_1 < b < \lambda_{l+1} < a = \widetilde{\theta}_{l+1} < \sigma_0$, then $\exists \delta_3 > 0$, $\forall \widetilde{a} \in [a-\delta_3, a]$, $\exists \widetilde{b} \in \mathbb{R}$, s.t., $\left(\widetilde{a},\widetilde{b}\right) \in F_m^0$.*

Proof. The conclusion is fully based on Lemma 5.28 and Lemma 5.34. □

Proof of Theorem 5.33. By Lemma 5.30, Lemma 5.32, Lemma 5.34 and Lemma 5.35, the consequence follows directly from an argument resembling that of Theorem 5.23. □

6. Critical groups of J at infinity and zero

In this section we present the computation of critical groups of J at zero and infinity, which are exploited as tools to deal with the problems on the existence of nontrivial solutions of (1.1).

6.1. Computation of $C_*(J,\infty)$

Let E be a Hilbert space and $\Phi : E \to \mathbb{R}$ be of class C^1. Recall the concept of critical groups of Φ at infinity introduced by [2]:

Definition 6.1. Suppose $\Phi(K_\Phi)$ is strictly bounded from below by $c \in \mathbb{R}$ and that Φ satisfies the (PS) condition. Then $C_q(\Phi,\infty) \cong H_q(E,\Phi^c)$, $q \geq 0$, is the q-th critical group of Φ at infinity. It is independent of the choice of c with (PS) condition.

Define

$$J_t(u) = J_t(u,a,b) = \frac{1}{2}\int\limits_{\mathbb{R}^N}\left(|\nabla u|^2 + V(x)u^2\right)dx - \frac{a}{2}\int\limits_{\mathbb{R}^N}\left(u^-\right)^2$$
$$-\frac{b}{2}\int\limits_{\mathbb{R}^N}\left(u^+\right)^2 - t\int\limits_{\mathbb{R}^N} G(x,u),$$

$t \in [0,1]$, $G(x,u) = \int_0^u g(x,s)\,ds$, and consider the following equation:

$$\begin{cases} -\Delta u + V(x)u = au^- + bu^+ + tg(x,u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{6.1}$$

Our main result concerning the computation of critical groups at infinity reads:

Theorem 6.2. *Suppose (f_1) and $(a,b) \notin \Sigma(A)$, then*

$$C_q(J,\infty) \cong C_q(J_t,\infty) \cong C_q(I,0),\ \forall t \in [0,1]. \tag{6.2}$$

To complete the proof of Theorem 6.2, we present some lemmas adapted to our needs.

Lemma 6.3. *Under the hypotheses of Theorem 6.2, there exists $M > 0$, s.t.*

$$\sup_{t\in[0,1]}\sup_{u\in K_{J_t}} \|u\|_m < M. \tag{6.3}$$

Proof. The conclusion follows directly from an argument analogous to that of Theorem 3.4.

Lemma 6.4. *Under the hypotheses of Lemma 6.3,*

$$\delta = \inf_{t\in[0,1],\ u\in H_m^1(\mathbb{R}^N)\setminus B_m(0,M)} \frac{\left\|J_t'(u)\right\|_m}{\|u\|_m} > 0, \tag{6.4}$$

where

$$J_t'(u) = u - A_m^{-1}\left[(a+m)u^- + (b+m)u^+ + tg(x,u)\right],$$
$$B_m(u,M) = \left\{w \in H_m^1 : \|w-u\|_m < M\right\}.$$

Proof. By way of negation, there exist $t_k \in [0,1]$, $u_k \in H_m^1\left(\mathbb{R}^N\right)\backslash B_m(0,M)$, s.t.

$$\frac{\left\|J_{t_k}'(u_k)\right\|_m}{\|u_k\|_m} \to 0, k \to +\infty \tag{6.5}$$

Repeating an argument analogous to Step two of the proof of Theorem 3.4, it follows that $\{u_k\}_{k=1}^{+\infty}$ is bounded in $H_m^1\left(\mathbb{R}^N\right)$ and $\left\|J_{t_k}'(u_k)\right\|_m \to 0$. Assume $t_k \to t_0$, $u_k \rightharpoonup u_0$ in $H_m^1\left(\mathbb{R}^N\right)$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N} \nabla u_0 \nabla\varphi + V(x)u_0\varphi \\ &= \int_{\mathbb{R}^N}\left(au_0^- + bu_0^+\right)\varphi + t_0\int_{\mathbb{R}^N} g(x,u_0)\varphi, \forall\varphi \in C_0^\infty\left(\mathbb{R}^N\right).\end{aligned} \tag{6.6}$$

Due to the density of $C_0^\infty\left(\mathbb{R}^N\right)$ in $H^1\left(\mathbb{R}^N\right)$, we yield

$$\begin{aligned}&\int_{\mathbb{R}^N} \nabla u_0 \nabla\varphi + V(x)u_0\varphi \\ &= \int_{\mathbb{R}^N}\left(au_0^- + bu_0^+\right)\varphi + t_0\int_{\mathbb{R}^N} g(x,u_0)\varphi, \forall\varphi \in H^1\left(\mathbb{R}^N\right),\end{aligned} \tag{6.7}$$

and this shows that u_0 is a weak solution of (6.1) for $t = t_0$, i.e.,

$$J_{t_0}'(u_0) = u_0 - A_m^{-1}\left[(a+m)u_0^- + (b+m)u_0^+ + t_0 g(x,u_0)\right] = 0. \tag{6.8}$$

Notice that

$$J_{t_k}'(u_k) = u_k - A_m^{-1}\left[(a+m)u_k^- + (b+m)u_k^+ + t_k g(x,u_k)\right] \to 0, \tag{6.9}$$

fully imitating the trick employed by the proof of Theorem 3.4, we get $u_k \to u_0$ in $H_m^1\left(\mathbb{R}^N\right)$, so $u_0 \in K_{J_{t_0}}\backslash B_m(0,M)$, which contradicts (6.3). The proof is complete. □

Lemma 6.5. *Under the hypothesis* (f_1), $\exists\widetilde{C} < 0$, $\forall t \in [0,1]$, *for* $u \in H_m^1\left(\mathbb{R}^N\right)$, *if* $J_t(u) \le \widetilde{C}$, *then* $\|u\|_m \ge M$.

Proof. Suppose the contrary. Take $C_k \to -\infty$ as $k \to +\infty$, then $\exists t_k \in [0,1]$ and $u_k \in E$, $J_{t_k}(u_k) \le C_k$, $\|u_k\|_m < M$. We just treat the case $\Lambda(a,b) + \beta > \mu_1$. Observe that

$$\begin{aligned} J_{t_k}(u_k) &\ge \frac{1}{2}\|u_k\|_m^2 - \frac{\Lambda(a,b)+\beta+m}{2}\int_{\mathbb{R}^N} u_k^2 \\ &\ge \frac{1}{2}\|u_k\|_m^2 - \frac{\Lambda(a,b)+\beta+m}{2(\mu_1+m)}\|u_k\|_m^2 \\ &= -\frac{\Lambda(a,b)+\beta-\mu_1}{2(\mu_1+m)}\|u_k\|_m^2 \\ &\ge -\frac{\Lambda(a,b)+\beta-\mu_1}{2(\mu_1+m)}M^2 > -\infty \end{aligned} \tag{6.10}$$

and this contradicts $J_{t_k}(u_k) \to -\infty$. The proof is complete. □

Lemma 6.6. *If $\widetilde{C} \ll -\frac{\Lambda(a,b)+\beta-\mu_1}{2(\mu_1+m)}M^2$, then $\partial_t J_t(w)\frac{J_t'(w)}{\|J_t'(w)\|_m^2}$ is local Lipschitz for $\forall u \in J_t^{\widetilde{C}} := \left\{u \in H_m^1 : J_t(u) \le \widetilde{C}\right\}$, $\forall t \in [0,1]$.*

Proof. Notice that

$$\partial_t J_t(w) = -\int_{\mathbb{R}^N} G(x,w),$$

we write $F_1(w) = \partial_t J_t(w)$, $F_2(w) = \frac{J_t'(w)}{\|J_t'(w)\|_m^2}$. As $\widetilde{C} \ll -\frac{\Lambda(a,b)+\beta-\mu_1}{2(\mu_1+m)}M^2$ indicates that $\|u\|_m \ge M$, by (6.4) we have

$$\inf_{t\in[0,1],\ w\in J_t^{\widetilde{C}}} \frac{\|J_t'(w)\|_m}{\|w\|_m} \ge \delta. \tag{6.11}$$

Therefore, there exist $R > 0$, s.t.

$$\inf_{t\in[0,1],\ w\in\overline{B_m(u,R)}} \frac{\|J_t'(w)\|_m}{\|w\|_m} \ge \frac{\delta}{2}, \tag{6.12}$$

and $\|w\|_m \ge \frac{M}{2}$ for $\forall w \in \overline{B_m(u,R)}$. This yields

$$\inf_{t\in[0,1],\ w\in\overline{B_m(u,R)}} \|J_t'(w)\|_m \ge \frac{M\delta}{4}. \tag{6.13}$$

Take $w_1, w_2 \in \overline{B_m(u,R)}$. Estimating crudely shows that for $\forall\varphi \in H_m^1(\mathbb{R}^N)$, $\|\varphi\|_m = 1$,

$$\left|\int_{\mathbb{R}^N} V\left(w_1-w_2\right)\varphi\right|$$
$$\le \|V_1\|_{L^p}\cdot\|\varphi\|_{L^{2q}}\cdot\|w_1-w_2\|_{L^{2q}}+\|V_2\|_{L^\infty}\cdot\|\varphi\|_{L^2}\cdot\|w_1-w_2\|_{L^2}$$
$$\le \widetilde{C}_1\|w_1-w_2\|_m \tag{6.14}$$

if $p>\frac{N}{2}\ge 2$, and

$$\left|\int_{\mathbb{R}^N} V(x)\left(w_1-w_2\right)\varphi\right|$$
$$\le \|V_1\|_{L^2}\cdot\|\varphi\|_{L^4}\cdot\|w_1-w_2\|_{L^4}+\|V_2\|_{L^\infty}\cdot\|\varphi\|_{L^2}\cdot\|w_1-w_2\|_{L^2}$$
$$\le \widetilde{C}_2\|w_1-w_2\|_m \tag{6.15}$$

if $p=2$ and $N\le 3$, $\frac{1}{p}+\frac{1}{q}=1$.

Thereby,

$$\left|\int_{\mathbb{R}^N} V(x)\left(w_1-w_2\right)\varphi\right|\le \widetilde{C}_3\|w_1-w_2\|_m, \tag{6.16}$$

$\widetilde{C}_3=\Lambda\left(\widetilde{C}_1,\widetilde{C}_2\right)$.

And also observe that for $\forall\varphi\in H_m^1\left(\mathbb{R}^N\right)$, $\|\varphi\|_m=1$,

$$\begin{aligned}\left|\int_\Omega\left(w_1^+-w_2^+\right)\varphi\right| &\le \left\|w_1^+-w_2^+\right\|_{L^2}\cdot\|\varphi\|_{L^2}\\ &=\left[\int_{w_1^+\ge w_2^+}\left(w_1^+-w_2^+\right)^2+\int_{w_2^+\ge w_1^+}\left(w_2^+-w_1^+\right)^2\right]^{\frac{1}{2}}\cdot\|\varphi\|_{L^2}\\ &\le\left[\int_{w_1^+\ge w_2^+}\left(w_1-w_2\right)^2+\int_{w_2^+\ge w_1^+}\left(w_1-w_2\right)^2\right]^{\frac{1}{2}}\cdot\|\varphi\|_{L^2}\\ &=2\|w_1-w_2\|_m.\end{aligned} \tag{6.17}$$

Proceeding along the same lines, we have

$$\left|\int_\Omega\left(w_1^--w_2^-\right)\varphi\right|=\left|\int_\Omega\left[(-w_2)^+-(-w_1)^+\right]\varphi\right|\le 2\|w_1-w_2\|_m. \tag{6.18}$$

Therefore,

$$\left\| J_t'(w_1) - J_t'(w_2) \right\|_m \leq \left(1 + \widetilde{C}_3 + 2|a+m| + 2|b+m| + 2\beta\right) \|w_1 - w_2\|_m . \tag{6.19}$$

Consequently,

$$\begin{aligned}
&\|F_2(w_1) - F_2(w_2)\|_m \\
&\leq \frac{\left\| J_t'(w_2) \right\|_m^2 \cdot \left\| J_t'(w_1) - J_t'(w_2) \right\|_m}{\left\| J_t'(w_1) \right\|_m^2 \cdot \left\| J_t'(w_2) \right\|_m^2} \\
&\quad + \frac{\left\| J_t'(w_1) \right\|_m + \left\| J_t'(w_2) \right\|_m}{\left\| J_t'(w_1) \right\|_m^2 \cdot \left\| J_t'(w_2) \right\|_m} \cdot \left\| J_t'(w_1) - J_t'(w_2) \right\|_m \\
&= \left(\frac{2}{\left\| J_t'(w_1) \right\|_m^2} + \frac{1}{\left\| J_t'(w_1) \right\|_m \cdot \left\| J_t'(w_2) \right\|_m} \right) \cdot \left\| J_t'(w_1) - J_t'(w_2) \right\|_m \\
&\leq \frac{48}{(M\delta)^2} \left\| J_t'(w_1) - J_t'(w_2) \right\|_m \leq \widetilde{C}_4 \|w_1 - w_2\|_m .
\end{aligned} \tag{6.20}$$

As

$$\begin{aligned}
|G(x, w_1) - G(x, w_2)| &= |g(x, w_2 + \theta(w_1 - w_2))| \cdot |w_1 - w_2| \\
&\leq \beta |w_2 + \theta(w_1 - w_2)| \cdot |w_1 - w_2| ,
\end{aligned} \tag{6.21}$$

$0 < \theta < 1$, we get

$$\begin{aligned}
|F_1(w_1) - F_1(w_2)| &\leq \beta \int_{\mathbb{R}^N} |w_2 + \theta(w_1 - w_2)| \cdot |w_1 - w_2| \\
&\leq 2\beta \|w_2\|_m \cdot \|w_1 - w_2\|_m + 2\beta \|w_1 - w_2\|_m^2 \\
&\leq 2\beta (3R + \|u\|_m) \cdot \|w_1 - w_2\|_m .
\end{aligned} \tag{6.22}$$

Hence,

$$\begin{aligned}
&\|F_1(w_1) F_2(w_1) - F_1(w_2) F_2(w_2)\|_m \\
&\leq \|(F_1(w_1) - F_1(w_2)) F_2(w_1)\|_m + \|(F_2(w_1) - F_2(w_2)) F_1(w_2)\|_m \\
&= |F_1(w_1) - F_1(w_2)| \cdot \|F_2(w_1)\|_m + |F_1(w_2)| \cdot \|F_2(w_1) - F_2(w_2)\|_m \\
&\leq \left[\frac{8}{M\delta} \beta (3R + \|u\|_m) + 2\widetilde{C}_4 \beta (3R + \|u\|_m)^2 \right] \cdot \|w_1 - w_2\|_m \\
&\leq \widetilde{C}_5 \|w_1 - w_2\|_m
\end{aligned} \tag{6.23}$$

and we arrive at the desired conclusion. □

Proof of Theorem 6.2. Consider the following equation on $H_m^1\left(\mathbb{R}^N\right)$,

$$\frac{d}{dt}\sigma\left(t,u\right)=-\partial_t J_t\left(\sigma\left(t,u\right)\right)\frac{J_t'\left(\sigma\left(t,u\right)\right)}{\left\|J_t'\left(\sigma\left(t,u\right)\right)\right\|_m^2},\sigma\left(0,u\right)=u,t\in[0,1] \tag{6.24}$$

(see [9]). Let $u\in I^{\widetilde{C}}=\left\{u\in H_m^1\left(\mathbb{R}^N\right):I\left(u\right)\le\widetilde{C}\right\}$, $\widetilde{C}\ll-\frac{\Lambda(a,b)+\beta-\mu_1}{2(\mu_1+m)}M^2$.

Since $\partial_t J_t\left(w\right)\frac{J_t'(w)}{\left\|J_t'(w)\right\|_m^2}$ is local Lipschitz for $\forall u\in J_t^{\widetilde{C}}$, $\forall t\in[0,1]$, there is a $t_0>0$ such that the solution of (6.24) exists for any initial value $u\in I^{\widetilde{C}}$ and $t\in[0,t_0]$.

Observe that

$$\frac{d}{dt}J_t\left(\sigma\left(t,u\right)\right)=\left\langle J_t'\left(\sigma\left(t,u\right)\right),\frac{d}{dt}\sigma\left(t,u\right)\right\rangle+\partial_t J_t\left(\sigma\left(t,u\right)\right)=0, \tag{6.25}$$

we have

$$J_t\left(\sigma\left(t,u\right)\right)\le\widetilde{C}\text{ if and only if }I\left(u\right)\le\widetilde{C}. \tag{6.26}$$

In view of the fact

$$\left|\partial_t J_t\left(u\right)\right|=\left|\int_{\mathbb{R}^N}G\left(x,u\right)\right|\le2\beta\left\|u\right\|_m^2, \tag{6.27}$$

consequently, by (6.11)

$$\left\|\partial_t J_t\left(\sigma\left(t,u\right)\right)\frac{J_t'\left(\sigma\left(t,u\right)\right)}{\left\|J_t'\left(\sigma\left(t,u\right)\right)\right\|_m^2}\right\|_m\le\frac{2\beta\left\|\sigma\left(t,u\right)\right\|_m^2}{\left\|J_t'\left(\sigma\left(t,u\right)\right)\right\|_m}\le\frac{2\beta}{\delta}\left\|\sigma\left(t,u\right)\right\|_m. \tag{6.28}$$

Notice that (6.24) derives

$$\sigma\left(t,u\right)-u=-\int_0^t\partial_s J_s\left(\sigma\left(s,u\right)\right)\frac{J_s'\left(\sigma\left(s,u\right)\right)}{\left\|J_s'\left(\sigma\left(s,u\right)\right)\right\|_m^2}ds. \tag{6.29}$$

Taking $\varphi=\frac{\sigma(t,u)}{\|\sigma(t,u)\|_m}$ yields

$$\left\langle\sigma\left(t,u\right),\varphi\right\rangle_m=\left\|\sigma\left(t,u\right)\right\|_m. \tag{6.30}$$

By Fubini theorem and (6.28) we obtain

$$\|\sigma(t,u)\|_m \le \|u\|_m + \int_0^t \frac{|\partial_s J_s(\sigma(s,u))|}{\left\|J_s'(\sigma(s,u))\right\|_m^2} \cdot \left\|J_s'(\sigma(s,u))\right\|_m ds$$
$$\le \|u\|_m + \frac{2\beta}{\delta}\int_0^t \|\sigma(s,u)\|_m ds \tag{6.31}$$

and by Gronwall inequality

$$\|\sigma(t,u)\|_m \le \|u\|_m \cdot e^{\frac{2\beta}{\delta}t}. \tag{6.32}$$

Repeat above procedure we deduce that (6.32) follows for $\forall t \in [0,1]$.

Now we define a map:

$$\Phi : u \mapsto \sigma(1,u),$$

which is a homeomorphism between the level sets of $I^{\widetilde{C}}$ and $J^{\widetilde{C}}$. Thereby,

$$H_q\left(J^{\widetilde{C}}\right) \cong H_q\left(I^{\widetilde{C}}\right) \tag{6.33}$$

and the proof is complete. □

6.2. Computation of $C_*(J,0)$

Also set $f(x,s) = a_0 s^- + b_0 s^+ + \widetilde{g}(x,s)$, $\widetilde{g}(x,s) = g(x,s) - (a_0 - a)s^- - (b_0 - b)s^+$, $g(x,0) = 0$ and assume $\lim\limits_{s\to 0} \frac{\widetilde{g}(x,s)}{s} = 0$, $\lim\limits_{s\to\infty} \frac{g(x,s)}{s} = 0$, uniformly with respect to $x \in \mathbb{R}^N$. To show our consequence in this subsection, we need the following stronger hypotheses:

(f_2) $\max\limits_{t\in[0,1]} \Lambda(a_t, b_t) \le \max\limits_{t\in[0,1]} (\Lambda(a_t,b_t) + t\beta) < \sigma_0$, $a_t = (1-t)a_0 + ta$, $b_t = (1-t)b_0 + tb$.

Define

$$\widetilde{J}_t(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right)dx - \frac{a_0}{2}\int_{\mathbb{R}^N} \left(u^-\right)^2$$
$$-\frac{b_0}{2}\int_{\mathbb{R}^N} \left(u^+\right)^2 - t\int_{\mathbb{R}^N} \widetilde{G}(x,u),$$

$t \in [0,1]$, $\widetilde{G}(x,u) = \int_0^u \widetilde{g}(x,s)\,ds$, and consider the following equation:

$$\begin{cases} -\Delta u + V(x)u = a_0 u^- + b_0 u^+ + t\widetilde{g}(x,u), & x \in \mathbb{R}^N, \\ u(x) \to 0, & \text{as } |x| \to +\infty. \end{cases} \tag{6.34}$$

Our main result concerning computation of critical groups of J at zero reads:

Lemma 6.7. *Let V be a real K–R potential and suppose (f_1), $(a_0, b_0) \notin \Sigma(A)$, $\Lambda(a_0, b_0) < \sigma_0$, then 0 is an isolated critical point of J. In addition,*

$$C_q(J, 0) \cong C_q\left(\widetilde{J}_t, 0\right) \cong C_q(I, 0), \forall t \in [0, 1]. \tag{6.35}$$

Remark 6.8. Note that the combination of (f_1) and $\Lambda(a_0, b_0) < \sigma_0$ alludes to (f_2).

Proof of Lemma 6.7. To show the first half of the proof, we verify that $\exists r > 0$, $\forall t \in [0, 1]$,

$$K_{\widetilde{J}_t} \cap B_m(0, r) = \{0\}, \tag{6.36}$$

where $K_{\widetilde{J}_t} := \left\{u \in H_m^1\left(\mathbb{R}^N\right) : \widetilde{J}_t'(u) = 0\right\}$. Suppose by contradiction, $\exists r_n > 0$, $r_n \to 0$, $\exists t_n \in [0, 1]$, $\exists u_n \in K_{\widetilde{J}_{t_n}} \cap B_m(0, r_n) \setminus \{0\}$, and thus $\|u_n\|_m \to 0$. Set $\widetilde{u}_n = \frac{u_n}{\|u_n\|_m}$. Hence,

$$\int_{\mathbb{R}^N} \nabla\widetilde{u}_n \nabla\varphi + V(x)\widetilde{u}_n\varphi - a_0\widetilde{u}_n^-\varphi - b_0\widetilde{u}_n^+\varphi - t_n\frac{\widetilde{g}(x, u_n)}{\|u_n\|_m}\varphi = 0, \forall\varphi \in H_m^1\left(\mathbb{R}^N\right). \tag{6.37}$$

Set $\|u_n\|_m = \rho_n$, and take $\delta_n > 0$, $\delta_n \to 0$, s.t. $\frac{\rho_n}{\delta_n} \to 0$. Now we claim that $\forall\varphi \in H_m^1\left(\mathbb{R}^N\right)$,

$$\lim_{n\to+\infty} \frac{\int_{\mathbb{R}^N} |\widetilde{g}(x, u_n)\varphi|}{\|u_n\|_m} = 0. \tag{6.38}$$

Indeed, observe that $\frac{f(x,s)}{s} \to a_0$ as $s \to 0^-$ and $\frac{f(x,s)}{s} \to b_0$ as $s \to 0^+$, then for $\forall\varepsilon > 0$, $\exists S > 0$, $\forall s \in \mathbb{R}$, $0 < |s| \le S$,

$$\max_{x\in\mathbb{R}^N} \left|\frac{\widetilde{g}(x, s)}{s}\right| < \frac{\varepsilon}{4\|\varphi\|_m}. \tag{6.39}$$

Clearly, $\exists N_1 \in \mathbb{N}$, $\forall n \ge N_1$, $\delta_n < S$. Therefore,

$$\begin{aligned} &\frac{\int_{|u_n|\le\delta_n} |\widetilde{g}(x, u_n)\varphi|}{\|u_n\|_m} \\ &\le \frac{\varepsilon}{4\|\varphi\|_m} \cdot \|\widetilde{u}_n\| \cdot \|\varphi\| \\ &\le \frac{\varepsilon}{4\|\varphi\|_m} \cdot 2\|\varphi\|_m = \frac{\varepsilon}{2}. \end{aligned} \tag{6.40}$$

On the other side, notice that $\widetilde{g}(x, s) = g(x, s) - (a_0 - a)s^- - (b_0 - b)s^+$, so there is a $C > 0$, for $\forall x \in \mathbb{R}^N$, $\forall s \in \mathbb{R}$, $|\widetilde{g}(x, s)| \le C|s|$.

If $N \ge 3$, picking $\widetilde{q} \in (2, 2^*) \Rightarrow \frac{2N}{N+2} < \widetilde{p} < 2$, $\frac{1}{\widetilde{p}} + \frac{1}{\widetilde{q}} = 1$. Choosing $\alpha \in \left(\frac{2}{N}, \frac{2}{N-2}\right) \Rightarrow 2 < (1 + \alpha)\widetilde{p} < 2^*$. As $\frac{\rho_n}{\delta_n} \to 0$, $\exists N_2 \in \mathbb{N}$, $\forall n \ge N_2$,

$$\begin{aligned}&\frac{\int_{|u_n|>\delta_n}|\widetilde{g}(x,u_n)\varphi|}{\|u_n\|_m}\\&\leq\frac{C\int_{|u_n|>\delta_n}|u_n|^{1+\alpha}\cdot|\varphi|}{\delta_n^{\alpha}\|u_n\|_m}\\&\leq\frac{C\left(\int_{\mathbb{R}^N}|u_n|^{(1+\alpha)\widetilde{p}}\right)^{\frac{1}{\widetilde{p}}}\cdot\left(\int_{\mathbb{R}^N}|\varphi|^{\widetilde{q}}\right)^{\frac{1}{\widetilde{q}}}}{\delta_n^{\alpha}\|u_n\|_m}\\&\leq\widetilde{C}\left(\frac{\rho_n}{\delta_n}\right)^{\alpha}\cdot\|\varphi\|_m<\frac{\varepsilon}{2}.\end{aligned}\tag{6.41}$$

Take $N=\Lambda(N_1,N_2)$, we have

$$\begin{aligned}&\frac{\int_{\mathbb{R}^N}|\widetilde{g}(x,u_n)\varphi|}{\|u_n\|_m}\\&\leq\frac{\int_{|u_n|\leq\delta_n}|\widetilde{g}(x,u_n)\varphi|}{\|u_n\|_m}+\frac{\int_{|u_n|>\delta_n}|\widetilde{g}(x,u_n)\varphi|}{\|u_n\|_m}\\&\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}\tag{6.42}$$

and the claim is thus proved.

Assume $\widetilde{u}_n\rightharpoonup u_0$ in $H_m^1(\mathbb{R}^N)$. A standard argument shows that u_0 is a weak solution of the following homogeneous problem:

$$\begin{cases}-\Delta u+V(x)u=a_0u^-+b_0u^+, & x\in\mathbb{R}^N,\\ u(x)\to 0, & \text{as } |x|\to+\infty.\end{cases}\tag{6.43}$$

Thanks to $(a_0,b_0)\notin\Sigma$, we get $u_0=0$. Observe that $\Lambda(a_0,b_0)<\sigma_0$ alludes to (f_2). Set $d_0=\max_{t\in[0,1]}(\Lambda(a_t,b_t)+t\beta)$. There is no harm supposing $d_0>\mu_1=\inf\sigma(A)$, s.t., $\sigma_{\text{dis}}(A)\cap[\mu_1,d_0]=\{\lambda_i\}_{i=1}^l$, $\lambda_1=\mu_1=\inf\sigma(A)<\lambda_2<\cdots<\lambda_l$.

It deduces from (6.37) that

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla\widetilde{u}_n|^2+V(x)|\widetilde{u}_n|^2&\leq\left(\Lambda\left(a_{t_n},b_{t_n}\right)+t_n\beta\right)\|\widetilde{u}_n\|_{L^2}^2\\&\leq d_0\|\widetilde{u}_n\|_{L^2}^2,\end{aligned}\tag{6.44}$$

i.e.,

$$\begin{aligned}&\int_{\mathbb{R}^N}\left|\nabla P_{N_l}\widetilde{u}_n\right|^2+V(x)\left(P_{N_l}\widetilde{u}_n\right)^2+\int_{\mathbb{R}^N}\left|\nabla P_{N_l^\perp}\widetilde{u}_n\right|^2+V(x)\left(P_{N_l^\perp}\widetilde{u}_n\right)^2\\&\leq d_0\left\|P_{N_l}\widetilde{u}_n\right\|_{L^2}^2+d_0\left\|P_{N_l^\perp}\widetilde{u}_n\right\|_{L^2}^2,\end{aligned}\tag{6.45}$$

yielding

$$\left\| P_{N_l^{\perp}} \widetilde{u}_n \right\|_{L^2}^2 \leq \frac{d_0 - \mu_1}{\theta_l - d_0} \left\| P_{N_l} \widetilde{u}_n \right\|_{L^2}^2 . \tag{6.46}$$

As $\widetilde{u}_n \rightharpoonup 0$ in $H_m^1\left(\mathbb{R}^N\right)$, we infer from (6.46) that

$$\widetilde{u}_n \to 0 \text{ in } L^2\left(\mathbb{R}^N\right), \tag{6.47}$$

and hence by (6.44),

$$1 \leq (d_0 + m) \left\| \widetilde{u}_n \right\|_{L^2}^2 \to 0. \tag{6.48}$$

A contradiction! Thus we get (6.36). In view of the hypotheses, $\widetilde{J}_t$ satisfies the (PS) condition for $\forall t \in [0, 1]$. Making use of Proposition 9.4, we arrive the conclusion. □

7. Four solutions theorem and some preliminaries on Morse theory

In this and next section we always assume that $V(x)$ is a real K–R potential and also make the following hypotheses:

(V_3) $V \in C^{N-2,\alpha}\left(\mathbb{R}^N\right)$ if $N \geq 3$ and $V \in C^1\left(\mathbb{R}^N\right)$ if $N = 1, 2$, $0 < \alpha < 1$;

(V_4) $\lambda_1 = \mu_1 = \inf \sigma(A) < \lambda_2 < \cdots < \lambda_l < \sigma_0$, $\sigma_{\text{dis}}(A) \cap [\mu_1, \sigma_0) = \{\lambda_i\}_{i=1}^{l}$;

(f_3) Let $f(x, s) = a_0 s^- + b_0 s^+ + \widetilde{g}(x, s)$, $\lim\limits_{s \to 0_-} \frac{f(x,s)}{s} = a_0$, $\lim\limits_{s \to 0_+} \frac{f(x,s)}{s} = b_0$, uniformly on $x \in \mathbb{R}^N$, and $\widetilde{g}(x, s) \in C^{N-2,\alpha}\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$ if $N \geq 3$, and $\widetilde{g}(x, s) \in C^1\left(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}\right)$ if $N = 1, 2$;

(f_4) $\lambda_k < \Gamma(a, b) \leq \Lambda(a, b) < \widetilde{\theta}_{k+1} = \Gamma\left(\mu_{d_{k+1}+1}, \sigma_0\right)$, $l > k \geq 3$;

(f_5) $\Lambda(a_0, b_0) < \lambda_1$;

(f_6) $f_s'(x, s) > \frac{f(x,s)}{s} > -m$, $\forall s \neq 0$, a.e. on $x \in \mathbb{R}^N$, where m is given by Section 3.

Rewrite J by

$$J(u) = \frac{1}{2} \|u\|_m^2 - \frac{m}{2} \|u\|_{L^2}^2 - \int_{\mathbb{R}^N} F(x, u).$$

Let $A_m = -\Delta + V + m$, if $m + \lambda_1 > 0$, then A_m is a positive definite self-adjoint operator with $D(A_m) = H^2\left(\mathbb{R}^N\right)$. According to the spectral resolution of self-adjoint operator we have

$$A_m = \int_0^{+\infty} \lambda d E_\lambda. \tag{7.1}$$

Lemma 7.1. *Suppose $\lambda_1 = \inf \sigma(A) < \sigma_0$. Under the hypothesis (V_2), if $u \in H_m^1$, $\|u\|_{L^2} = 1$, then*

(1) $\langle u, v \rangle_m = \langle (m + \lambda_1) u, v \rangle_{L^2}$ iff $\langle u, u \rangle_m = m + \lambda_1$, for $\forall v \in H_m^1$;

(2) The dimension of solution space E_1 of the following linear equation

$$\langle u, v \rangle_m = \langle (m + \lambda_1) u, v \rangle_{L^2}, \forall v \in H_m^1, \tag{7.2}$$

is 1*, and so we can choose* $\psi_1 > 0$*, the corresponding eigenvector of* λ_1*, such that* $E_1 = N_1 = \operatorname{span}\{\psi_1\}$.

Proof. As $H_m^1 = H^1$, decompose $H_m^1 = N_l \bigoplus M_l$. We just deal here with the "if" part since the "only if" part is trivial. Set $u = v + w$, $\|u\|_{L^2} = 1$, $v \in N_l$, $w \in M_l$. We claim that $u \in N_l$. In fact, if $w \neq 0$, then $\langle A_m v, w\rangle_{L^2} = 0$. Therefore, by using Lemma 2.11, we have

$$\begin{aligned}\lambda_1 + m &= \langle A_m v, v\rangle_{L^2} + \langle w, w\rangle_m + 2\langle A_m v, w\rangle_{L^2} \\ &\geq (\lambda_1 + m)\|v\|_{L^2}^2 + (\sigma_0 + m)\|w\|_{L^2}^2 > \lambda_1 + m. \end{aligned} \tag{7.3}$$

A paradox! A similar way shows $u \in N_1$ if $v \in N_1$, $w \in M_1$. The claim (1) is thus proved and (2) follows directly from Theorem XIII.48 of [33], ending the proof. □

It is clearly apparent that $J \notin C^2\left(H_m^1, \mathbb{R}\right)$ as $f(x, s)$ has jumping nonlinearity at 0, so we need some preliminaries on Morse theory for this kind of functional.

In order to use Morse theory for jumping nonlinear problems, we present the following splitting theorem and shifting theorem and they were essentially given in [20] and [21].

Let X and Y be Banach spaces and U be a subset of X. Recall that $F : U \to Y$ is said to be a strictly Fréchet differentiable at $x_0 \in U$ if there exists $A \in L(X, Y)$ such that

$$\lim_{x_1,\, x_2 \to x_0} \frac{\|F(x_2) - F(x_1) - A(x_2 - x_1)\|}{\|x_2 - x_1\|} = 0. \tag{7.4}$$

Compared to a splitting theorem yielded by [21], the following statement shows a little variation version but the proof is almost the same.

Proposition 7.2. *(Splitting theorem) Let* E *be a Hilbert space,* $J \in C^1(E, \mathbb{R})$ *and* $u_0 \in E$ *be an isolated critical point of* J*. Assume*

(i) $\nabla J : E \to E$ *is Lipschitz continuous, and* $\nabla J(u)$ *is strictly Fréchet differentiable at* u_0*, and* $\widetilde{A} = J''(u_0)$ *is a Fredholm operator with index* 0*;*

(ii) With $N = \ker \widetilde{A}$ *and* $N^{\perp_E}$ *being the orthogonal complement of* N *in* E*, for* $\forall v \in N$*, there exists a dense subset* D_v *of* $N^{\perp_E}$ *such that* $\lambda w_1 + (1-\lambda) w_2 \in D_v$ *for a.e.* $\lambda \in [0, 1]$*, if* $w_1, w_2 \in D_v$*.* $J''(u_0 + v + w)$ *exists for* $\forall (v, w) \in Y_N$*, and* $J''(u_0 + \cdot + \cdot) : Y_N \to L(E, E)$ *is continuous, where* $Y_N = \{(v, w) : v \in N, w \in D_v\}$*. Then there exist a ball* $B(u_0, \delta)$ *in* E *centered at* u_0 *and with radius* $\delta > 0$*, a* u_0*-preserving local homeomorphism* h *from* $B(u_0, \delta)$ *to* $h(B(u_0, \delta)) \subset E$ *and a Lipschitz continuous map* $g : B(u_0, \delta) \cap N \to N^{\perp_E}$ *such that*

$$J(h(u)) = \frac{1}{2}\left\langle \widetilde{A} w, w\right\rangle + J(u_0 + v + g(v)), \tag{7.5}$$

$u - u_0 = v + w,\ v \in N,\ w \in N^{\perp_E}.$

Proposition 7.3. *(Shifting theorem) Under the hypotheses of Proposition 7.2, if there is an orthogonal decomposition* $N^{\perp_E} = W_1 \bigoplus_E W_2$*, such that* $j = \dim W_1 < +\infty$ *and*

(i) $P_2 J'(u_0 + v + w_1 + \cdot) : W_2 \to W_2$ *is* μ*-monotone for all* $(v, w_1) \in N \bigoplus_E W_1$*;*

(ii) $-P_1 J'(u_0 + v + \cdot + w_2) : W_1 \to W_1$ *is* μ*-monotone for all* $(v, w_2) \in N \bigoplus_E W_2$*, where* $\bigoplus_E$ *denotes decomposition into direct sum in* E*, and* $P_i : E \to W_i$ *is the projection onto* W_i*,* $i = 1, 2$*.*

Denote $\widetilde{J}(v) = J(u_0 + v + g(v))$*, then*

$$C_q(J, u_0) \cong C_{q-j}\left(\widetilde{J}, 0\right). \tag{7.6}$$

Lemma 7.4. *Under the hypotheses* (f_1)*,* (f_3)*, then* $\nabla J(u) : H_m^1 \to H_m^1$ *is Lipschitz continuous.*

Proof. We just need to verify that

$$A_m^{-1}(f(x, u)) : H_m^1 \to H_m^1 \tag{7.7}$$

is Lipschitz continuous, i.e., there exists $l > 0$, s.t.

$$\left\| A_m^{-1}(f(x, u_1) - f(x, u_2)) \right\|_m \leq l \|u_1 - u_2\|_m, \forall u_1, u_2 \in H_m^1. \tag{7.8}$$

(f_1) indicates that $\exists C > 0$,

$$|f(x, s_1) - f(x, s_2)| < C|s_1 - s_2|, \forall s_1, s_2 \in \mathbb{R}, x \in \mathbb{R}^N. \tag{7.9}$$

Therefore,

$$\int_{\mathbb{R}^N} |f(x, u_1) - f(x, u_2)|^2 \leq C^2 \int_{\mathbb{R}^N} |u_1 - u_2|^2. \tag{7.10}$$

Notice that $A_m : L^2 \to L^2$ is a self-adjoint operator with $D(A_m) = H^2\left(\mathbb{R}^N\right)$, then $T(A_m) = \int_{-\infty}^{+\infty} T(\lambda)\, dE_\lambda$ is also a self-adjoint operator, where $T: \mathbb{R}^1 \to \mathbb{R}^1$ is any bounded Borel measurable function. Especially, $A_m^{-1} : L^2 \to L^2$ is self-adjoint.

In view of the definition, for $\forall u \in L^2$, $A_m^{-1}u \in H^2$ as $\inf \sigma(A_m) > 0$. Hence,

$$\begin{aligned}
\left\| A_m^{-1}u \right\|_m^2 &= \left\langle A_m^{-1}u, A_m^{-1}u \right\rangle_m \\
&= \int_{\mathbb{R}^N} \left| \nabla A_m^{-1}u \right|^2 + (V + m)\left(A_m^{-1}u\right)^2 \\
&= \int_{\mathbb{R}^N} A_m A_m^{-1}u \cdot A_m^{-1}u \\
&= \left\langle A_m^{-1}u, u \right\rangle_{L^2}.
\end{aligned} \tag{7.11}$$

Set $\alpha_0 = m + \lambda_1$. Observe that for $\forall \lambda \in \sigma(A_m)$, $\alpha_0 \leq \lambda < +\infty \Rightarrow 0 < \frac{1}{\lambda} \leq \frac{1}{\alpha_0}$. Set $\mu = \frac{1}{\lambda}$, then $\sigma\left(A_m^{-1}\right) \subset \left[0, \frac{1}{\alpha_0}\right]$. Due to the fact $A_m^{-1}u = \int_{-\infty}^{+\infty} \mu\, dE_\mu u$, we have

$$\left\langle A_m^{-1}u, u\right\rangle_{L^2} = \int_{-\infty}^{+\infty} \mu d\left\langle E_\mu u, u\right\rangle_{L^2}$$
$$= \int_0^{\frac{1}{\alpha_0}} \mu d\left\langle E_\mu u, u\right\rangle_{L^2}$$
$$\leq \frac{1}{\alpha_0}\|u\|_{L^2}^2. \tag{7.12}$$

(7.11) along with (7.12) yields

$$\left\|A_m^{-1}u\right\|_m \leq \alpha_0^{-\frac{1}{2}}\|u\|_{L^2}. \tag{7.13}$$

(7.13) implies that $A_m^{-1}: L^2 \to H_m^1$ is a bounded linear operator. The combination of (7.10) and (7.13) derives

$$\left\|A_m^{-1}\left(f(x,u_1) - f(x,u_2)\right)\right\|_m$$
$$\leq \alpha_0^{-\frac{1}{2}}\|f(x,u_1) - f(x,u_2)\|_{L^2}$$
$$\leq C\alpha_0^{-\frac{1}{2}}\|u_1 - u_2\|_{L^2} \leq C\alpha_0^{-1}\|u_1 - u_2\|_m. \tag{7.14}$$

The proof is complete. □

Define

$$D_0\left(\mathbb{R}^N\right) = \left\{u \in C^N\left(\mathbb{R}^N\right) : |u(x)| + |\nabla u(x)| \neq 0, u(x) \to 0 \text{ as } |x| \to +\infty\right\};$$
$$D_1\left(\mathbb{R}^N\right) = \left\{u \in C^N\left(\mathbb{R}^N\right) : \text{mes}\left\{x \in \mathbb{R}^N; u(x) = 0\right\} = 0\right\};$$
$$D\left(\mathbb{R}^N\right) = \left\{u \in H_m^1\left(\mathbb{R}^N\right) : \text{mes}\left\{x \in \mathbb{R}^N; u(x) = 0\right\} = 0\right\}.$$

Lemma 7.5. *Suppose* (V_3), (V_4), (f_1), (f_3) *hold, then*

(1) *for any solution* u_0 *of* (1.1), $u_0 \in D\left(\mathbb{R}^N\right)$;

(2) $\nabla J(\widetilde{u})$ *is strictly Fréchet differentiable in* H_m^1 *for* $\forall \widetilde{u} \in D\left(\mathbb{R}^N\right)$ *and* $J''(\widetilde{u})$ *is a bounded operator from* H_m^1 *into* H_m^1;

(3) $J''(u_0)$ *is a Fredholm operator with index zero.*

Proof. Replace

$$-\Delta u_0 + V(x)u_0 = f(x,u_0), \tag{7.15}$$

with

$$-\Delta u_0 + \widetilde{V}(x) u_0 = 0, \tag{7.16}$$

where

$$\widetilde{V}(x) = \begin{cases} V(x), & x \in \mathbb{R}^N, u_0(x) = 0, \\ V(x) - \frac{f(x, u_0(x))}{u_0(x)}, & x \in \mathbb{R}^N, u_0(x) \neq 0. \end{cases} \tag{7.17}$$

As $\widetilde{V}(x) \in L_{loc}^{\frac{N}{2}}(\mathbb{R}^N)$, by the unique continuation property of solutions of Schrödinger equation (see [15], [17], [18], [30]), we get $u_0 \in D(\mathbb{R}^N)$.

Let $I^{\pm}(u) = u^{\pm}$, $u \in H_m^1$, for any $\widetilde{u} \in D(\mathbb{R}^N)$, define

$$A_{\widetilde{u}}^{\pm} u(x) = \begin{cases} u(x), & x \in R_{\pm}^N(\widetilde{u}), \\ 0, & x \in R_{\mp}^N(\widetilde{u}) \cup R_0^N(\widetilde{u}), \end{cases} \tag{7.18}$$

where

$$R_+^N(\widetilde{u}) = \left\{x \in \mathbb{R}^N : \widetilde{u}(x) > 0\right\};$$

$$R_-^N(\widetilde{u}) = \left\{x \in \mathbb{R}^N : \widetilde{u}(x) < 0\right\};$$

$$R_0^N(\widetilde{u}) = \left\{x \in \mathbb{R}^N : \widetilde{u}(x) = 0\right\}.$$

It is easy to check that $A_{\widetilde{u}}^{\pm} \in L\left(H_m^1, L^q\left(\mathbb{R}^N\right)\right)$, $\forall q \in (1, 2^*)$, where $L(X, Y)$ denotes the space of all bounded linear operators from Banach space X into Banach space Y.

First, we show that $\left(I^{\pm}\right)' : D\left(\mathbb{R}^N\right) \to L\left(H_m^1, L^q\left(\mathbb{R}^N\right)\right)$, $\forall q \in (1, 2^*)$, and the Fréchet derivative $\left(I^{\pm}\right)'(\widetilde{u}) = A_{\widetilde{u}}^{\pm}$. For $u \in H_m^1$, denoting

$$E(u, \widetilde{u}) = \left(R_-^N(\widetilde{u} + u) \cap R_+^N(\widetilde{u})\right) \cup \left(R_+^N(\widetilde{u} + u) \cap R_-^N(\widetilde{u})\right). \tag{7.19}$$

If $\operatorname{mes}(E(u, \widetilde{u})) < +\infty$, then

$$\begin{aligned} &\left\|(\widetilde{u} + u)^{\pm} - \widetilde{u}^{\pm} - A_{\widetilde{u}}^{\pm} u\right\|_{L^q} \\ &= \left(\int_{E(u,\widetilde{u})} |\widetilde{u} + u|^q\right)^{\frac{1}{q}} \\ &\leq \left(\int_{E(u,\widetilde{u})} |u|^q\right)^{\frac{1}{q}} \\ &\leq \left(\operatorname{mes}(E(u, \widetilde{u}))\right)^{\frac{1}{q} - \frac{1}{2^*}} \|u\|_{L^{2^*}} \leq C \|u\|_m. \end{aligned} \tag{7.20}$$

It suffices to verify that

$$\operatorname{mes}\left(E\left(u,\widetilde{u}\right)\right)\to 0 \text{ as } \|u\|_m\to 0. \tag{7.21}$$

As a matter of fact, notice that

$$\lim_{n\to+\infty}\operatorname{mes}\left\{x\in\mathbb{R}^N:0<\widetilde{u}\left(x\right)\le\frac{1}{n}\right\}=0, \tag{7.22}$$

hence, $\forall\varepsilon>0$, $\exists N\in\mathbb{N}$, $\forall n\ge N$,

$$\operatorname{mes}\left\{x\in\mathbb{R}^N:0<\widetilde{u}\left(x\right)\le\frac{1}{n}\right\}<\frac{\varepsilon}{2}. \tag{7.23}$$

For fixed $n_0\in\mathbb{N}$, $n_0\ge N$,

$$\begin{aligned}&\operatorname{mes}\left(R_-^N\left(\widetilde{u}+u\right)\cap R_+^N\left(\widetilde{u}-\frac{1}{n_0}\right)\right)\\&=\int\limits_{R_-^N\left(\widetilde{u}+u\right)\cap R_+^N\left(\widetilde{u}-\frac{1}{n_0}\right)}1\\&\le n_0^{2^*}\int\limits_{R_-^N\left(\widetilde{u}+u\right)\cap R_+^N\left(\widetilde{u}-\frac{1}{n_0}\right)}|u|^{2^*}\\&\le Cn_0^{2^*}\|u\|_m^{2^*},\end{aligned} \tag{7.24}$$

indicating that $\exists\delta=\left(\frac{\varepsilon}{2C}\right)^{\frac{1}{2^*}}n_0^{-1}>0$, if $\|u\|_m\le\delta$, then

$$\operatorname{mes}\left(R_-^N\left(\widetilde{u}+u\right)\cap R_+^N\left(\widetilde{u}-\frac{1}{n_0}\right)\right)\le\frac{\varepsilon}{2}. \tag{7.25}$$

Consequently, combining (7.23) and (7.25),

$$\begin{aligned}&\operatorname{mes}\left(R_-^N\left(\widetilde{u}+u\right)\cap R_+^N\left(\widetilde{u}\right)\right)\\&\le\operatorname{mes}\left(R_-^N\left(\widetilde{u}+u\right)\cap R_+^N\left(\widetilde{u}-\frac{1}{n_0}\right)\right)\\&\quad+\operatorname{mes}\left(R_-^N\left(\widetilde{u}+u\right)\cap\left\{x\in\mathbb{R}^N:0<\widetilde{u}\left(x\right)\le\frac{1}{n_0}\right\}\right)\\&\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned} \tag{7.26}$$

Proceeding along the same lines, we have

$$\operatorname{mes}\left(R_+^N\left(\widetilde{u}+u\right)\cap R_-^N\left(\widetilde{u}\right)\right)\to 0 \text{ as } \|u\|_m\to 0. \tag{7.27}$$

Therefore, (7.21) is valid and $\left(I^\pm\right)'\left(\widetilde{u}\right)=A_{\widetilde{u}}^\pm$ for $\widetilde{u}\in D\left(\mathbb{R}^N\right)$.

For $\widetilde{u}, \widetilde{u}+u \in D\left(\mathbb{R}^N\right)$,

$$\begin{aligned}
&\left\| A^{\pm}_{\widetilde{u}+u} - A^{\pm}_{\widetilde{u}} \right\|_{L\left(L^{2^*},L^2\right)} \\
&= \sup_{\|v\|_{2^*}=1} \left\| \left(A^{\pm}_{\widetilde{u}+u} - A^{\pm}_{\widetilde{u}} \right) v \right\|_{L^2} \\
&= \sup_{\|v\|_{2^*}=1} \left(\int_{E(u,\widetilde{u})} |v|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\operatorname{mes} E\left(u,\widetilde{u}\right)\right)^{\frac{1}{2}-\frac{1}{2^*}}, \qquad (7.28)
\end{aligned}$$

yielding $A^{\pm}_{\widetilde{u}} \in L\left(H^1_m, L^2\right)$. The combination of (7.20) and (7.21) obtains

$$J''\left(\widetilde{u}\right) = id - A_m^{-1}\left(aA^{-}_{\widetilde{u}} + bA^{+}_{\widetilde{u}} + g'_s\left(x,\widetilde{u}\right) + m\right). \qquad (7.29)$$

This together with (7.13) and $aA^{-}_{\widetilde{u}} + bA^{+}_{\widetilde{u}} + g'_s\left(x,\widetilde{u}\right) + m \in L\left(H^1_m, L^2\right)$ shows that $J''\left(\widetilde{u}\right) : H^1_m \to H^1_m$ is a bounded operator. That's the precise statement (2).

Observe that $(f_1) + (f_3) \Rightarrow \sigma_0 > \Lambda(a,b) + \beta > f'_s(x,s)$ for $\forall s \in \mathbb{R}\backslash\{0\}$, a.e. on $x \in \mathbb{R}^N$. Using (V_4), (f_1), (f_3), it follows that if u_0 is a solution of (1.1), for $\forall v \in H^2$,

$$\left\langle \left(-\Delta + V - f'_s\left(x,u_0\right)\right) v, v \right\rangle_{L^2} \geq \left\langle \left(-\Delta + V - \Lambda(a,b) - \beta\right) v, v \right\rangle_{L^2}, \qquad (7.30)$$

then by Proposition 2.7

$$\inf \sigma_{\text{ess}}\left(A_m J''\left(u_0\right)\right) \geq \inf \sigma_{\text{ess}}\left(A_0\right) = \sigma_0 - \Lambda(a,b) - \beta, \qquad (7.31)$$

dropping hints that $\sigma_{\text{ess}}\left(A_m J''\left(u_0\right)\right) \subset \left[\sigma_0 - \Lambda(a,b) - \beta, +\infty\right)$ and thus $m\left(u_0\right) + N\left(u_0\right) < +\infty$, where $A_0 = -\Delta + V - \Lambda(a,b) - \beta$, $m\left(u_0\right)$ denotes Morse index of J at u_0 and $N\left(u_0\right) = \dim\ker J''\left(u_0\right)$. Since $J''\left(u_0\right)$ is self-adjoint, it concludes claim (3) and the lemma is proved. □

In what follows, we set $E = H^1_m$. Now we present a crucial consequence stated as follows:

Lemma 7.6. *Let u_0 be a solution of (1.1), $\widetilde{A} = J''\left(u_0\right)$, $N = \ker \widetilde{A}$, $E = N \bigoplus_E N^{\perp_E}$. Under the hypotheses of Lemma 7.5, then*

(1) *For any $v \in N$, there exists a dense subset D_v of $N^{\perp_E}$ such that $\lambda w_1 + (1-\lambda) w_2 \in D_v$ for a.e. $\lambda \in [0,1]$, if $w_1, w_2 \in D_v$;*

(2) *$J''\left(u_0 + v + w\right)$ exists for any $v \in N$ and $w \in D_v$;*

(3) *$J''\left(u_0 + v + w\right) : Y_N \to L(E,E)$ is continuous, $Y_N = \{(v,w) : v \in N, w \in D_v\}$.*

Proof. Define

$$W = \left\{ w \in C^N\left(\mathbb{R}^N\right) \cap E : \langle w, v \rangle_m = 0, \forall v \in N \right\};$$

$$D_v = \left\{ w \in W : \operatorname{mes}\left\{ x \in \mathbb{R}^N : u_0(x) + v(x) + w(x) = 0 \right\} = 0 \right\}.$$

We claim that $\lambda w_1 + (1-\lambda) w_2 \in D_v$ for a.e. $\lambda \in [0,1]$ if $w_1, w_2 \in D_v$. Denote $B = \left\{x \in \mathbb{R}^N : u_0(x) \neq 0, u_0(x) + v(x) + w_2(x) \neq 0\right\}$. By (V_3), (f_3) and regularity of solutions for Schrödinger equation (see [22]) we get $u_0 \in C^N(B)$. As $w_2 \in D_v$, in view of the definition of B, we have $\text{mes}\left(\mathbb{R}^N \backslash B\right) = 0$. Notice that $u_0(x) + v(x) + w_1(x) \in C^N(B)$, by Sard theorem, for a.e. $\mu \in \mathbb{R}$ if $\frac{u_0(x)+v(x)+w_1(x)}{u_0(x)+v(x)+w_2(x)} = \mu$ at some $x \in B$, then

$$\begin{aligned}
0 &\neq (u_0(x) + v(x) + w_2(x)) \cdot \nabla \left(\frac{u_0(x) + v(x) + w_1(x)}{u_0(x) + v(x) + w_2(x)}\right) \\
&= \nabla (u_0(x) + v(x) + w_1(x)) \\
&\quad - \frac{u_0(x) + v(x) + w_1(x)}{u_0(x) + v(x) + w_2(x)} \cdot \nabla (u_0(x) + v(x) + w_2(x)) \\
&= \nabla F(x),
\end{aligned} \tag{7.32}$$

and this shows $\text{mes}\{x \in B : F(x) = 0\} = 0$, where

$$F(x) = u_0(x) + v(x) + w_1(x) - \mu(u_0(x) + v(x) + w_2(x)).$$

Let $D_v(B) = \{w \in W : \text{mes}\{x \in B : u_0(x) + v(x) + w(x) = 0\} = 0\}$. Take $\mu = \frac{\lambda - 1}{\lambda}$, then we get

$$\begin{aligned}
&\lambda(u_0(x) + v(x) + w_1(x)) + (1-\lambda)(u_0(x) + v(x) + w_2(x)) \\
&= u_0(x) + v(x) + \lambda w_1(x) + (1-\lambda) w_2(x),
\end{aligned} \tag{7.33}$$

and $\lambda w_1(x) + (1-\lambda) w_2(x) \in D_v(B)$. Since $\text{mes}\left(\mathbb{R}^N \backslash B\right) = 0$, claim (1) is verified. In view of the definition of D_v, we arrive at conclusion (2) via Lemma 7.5.

For fixed $v_0 \in N$, $w_0 \in D_{v_0}$, due to the facts that $E \hookrightarrow L^{2^*}$ and $A_m^{-1} \in L\left(L^2, E\right)$, by (7.28) we get

$$\left\| A_m^{-1}\left(A^{\pm}_{u_0+v+w} - A^{\pm}_{u_0+v_0+w_0}\right)\right\|_{L(E,E)} \to 0 \tag{7.34}$$

as $v \to v_0$, $w \to w_0$, $v \in N$, $w \in D_v$. Consequently (7.34) derives assertion (3). The proof is complete. □

Lemma 7.7. *Suppose that u_0 is a solution of (1.1) and $m(u_0) = 0$. Under the hypotheses of Lemma 7.5, if $\lambda = 1$ is an eigenvalue of the following weighted eigenequation*

$$\begin{cases} (-\Delta + V + m)\varphi = \lambda\left(f_s'(x, u_0) + m\right)\varphi, & x \in \mathbb{R}^N, \\ \varphi(x) \to 0, & |x| \to +\infty, \end{cases} \tag{7.35}$$

then 1 is the first eigenvalue of (7.35) and $\dim E_1(u_0) = 1$, where $E_1(u_0)$ is the eigenspace corresponding to $\lambda = 1$.

Proof. By Lemma 7.5 we have $u_0 \in D\left(\mathbb{R}^N\right)$, and combining this with (f_1) and (f_3), $f_s'(x, u_0) \in L^\infty\left(\mathbb{R}^N\right)$, indicating that V^* is a K–R potential, $V^*(x) = V(x) - f_s'(x, u_0)$. Based on the hypothesis, it admits of no doubt that $\eta = 0$ is the first eigenvalue of the following linear problem

$$\begin{cases} (-\Delta + V^*)\varphi = \eta\varphi, & x \in \mathbb{R}^N, \\ \varphi(x) \to 0, & |x| \to +\infty, \end{cases} \tag{7.36}$$

and invoking Lemma 7.1 the assertion follows. □

Lemma 7.8. *Suppose that u_0 is a mountain-pass type solution of (1.1). Under the hypotheses of Lemma 7.5, then*

$$C_q(J, u_0) \cong \delta_{q1} G. \tag{7.37}$$

Proof. By Lemma 7.4, Lemma 7.5 and Lemma 7.6, we get Proposition 7.3 and $m(u_0) < +\infty$. Set $j = m(u_0)$. Consequently,

$$C_q(J, u_0) \cong C_{q-j}\left(\widetilde{J}, u_0\right). \tag{7.38}$$

$\widetilde{J}(v) = J(u_0 + v + g(v))$. If $j = 1$, $C_0\left(\widetilde{J}, u_0\right) \neq 0$, indicating that 0 is a local minimizer of $\widetilde{J}$ and thus we yield (7.37) by (7.38). If $j = 0$, as $C_1\left(\widetilde{J}, u_0\right) \neq 0$, it deduces from Lemma 7.7 that $N(u_0) = 1$ and then 0 is a local maximizer of $\widetilde{J}$. Again by (7.38) we also derive (7.37), ending the proof. □

8. Proof of Theorem 1.1

8.1. A weak maximum principle for $\mathbb{R}^N$

We first recall some concepts and classical results. Consider the elliptic operator L of the form

$$Lu = -D_i\left(a^{ij}(x) D_j u + b^i(x) u\right) + c^i(x) D_i u + d(x) u, a^{ij} = a^{ji}, \tag{8.1}$$

and coefficients a^{ij}, b^i, c^i, d $(i, j = 1, \cdots, N)$ are assumed to be measurable functions on a domain $\Omega \subset \mathbb{R}^N$. If u is weakly differentiable, and $a^{ij} D_j u + b^i u$, $c^i D_i u + du$ are locally integrable, then, in a weak or generalized sense, u is said to satisfy $Lu = 0$ $(\geq 0, \leq 0)$ respectively in Ω, i.e.,

$$\varpi(u, v) = \int_\Omega \left[-D_i\left(a^{ij} D_j u + b^i u\right) v + \left(c^i D_i u + du\right) v\right] dx = 0, (\geq 0, \leq 0) \tag{8.2}$$

for $\forall v \in C_0^1(\Omega)$, $v \geq 0$ on Ω (see [14]).

Proposition 8.1. *(see [14]) Let $u \in W^{1,2}(\Omega)$ satisfy $Lu \geq 0$ in Ω. Then*

$$\inf_{x \in \Omega} u \geq \inf_{x \in \partial\Omega} u^-. \tag{8.3}$$

From Proposition 8.1 we derive the following fact:

Lemma 8.2. *Let $u \in H^1\left(\mathbb{R}^N\right)$ satisfy $Lu \geq 0$ in $\mathbb{R}^N$. Then $u \geq 0$ on $\mathbb{R}^N$.*

Proof. Suppose to contrary that $\exists x_0 \in \mathbb{R}^N$, $u(x_0) < 0$. Set $\alpha = -u(x_0)$. Pick $R > 0$, $\Omega = B(0, R)$, $x_0 \in \Omega$. By Proposition 8.1,

$$\inf_{x\in\partial\Omega} u^- \leq -\alpha, \tag{8.4}$$

which implies that for $\varepsilon = \frac{\alpha}{2}$, $\exists x^* \in \partial\Omega$,

$$u\left(x^*\right) \leq -\alpha + \frac{\alpha}{2} = -\frac{\alpha}{2}. \tag{8.5}$$

Choose $R_n \to +\infty$, then $\exists x_n^* \in \partial\Omega_n$,

$$u\left(x_n^*\right) \leq -\frac{\alpha}{2}, \tag{8.6}$$

$\Omega_n = B(0, R_n)$. However, this contradicts $u\left(x_n^*\right) \to 0$ since $u \in H^1\left(\mathbb{R}^N\right)$. □

8.2. *Hilbert–Riemann manifold* N

Let

$$\begin{aligned}
\mathrm{N} &= \left\{u \in E\setminus\{0\} : \left\langle J'(u), u\right\rangle_m = 0\right\};\\
S_1 &= \left\{\widehat{u} \in E : \|\widehat{u}\|_m = 1\right\};\\
\widehat{S} &= \left\{\widehat{u} \in S_1 : \widehat{u} = \frac{u}{\|u\|_m}, u \in \mathrm{N}\right\}.
\end{aligned}$$

Lemma 8.3. *Under the hypotheses* (f_3), (f_5), (f_6),
(1) $\widehat{S}$ *is an open set in* S_1;
(2) N *is a Hilbert–Riemann manifold without boundary.*

Proof. We first claim that for each $\widehat{u} \in \widehat{S}$ there exists a unique $t(\widehat{u}) > 0$ such that $t(\widehat{u})\widehat{u} \in \mathrm{N}$, and $t(\widehat{u})\widehat{u} : \widehat{S} \to \mathrm{N}$ is a differentiable homeomorphism. Notice that for $u \in N$, set $u = t\widehat{u}$, $t > 0$, $\widehat{u} \in S_1$, then we have

$$\int_{\mathbb{R}^N} |\nabla\widehat{u}|^2 + (V + m)\widehat{u}^2 = \frac{1}{t}\int_{\mathbb{R}^N} g_m(x, t\widehat{u})\widehat{u}, \tag{8.7}$$

$g_m(x, u) = f(x, u) + mu$. By (f_6), the right hand side of (8.7) is an increasing function of t. Therefore there exists a unique $t^* > 0$ such that $t^*\widehat{u} \in \mathrm{N}$. Let $h(t, \widehat{u}) = \left\langle J'(t\widehat{u}), t\widehat{u}\right\rangle_m$. We show that $h'_t(t, \widehat{u}) < 0$ if $h(t, \widehat{u}) = 0$. Clearly, $h(t, \widehat{u}) = 0$ if and only if (8.7) follows. Again by (f_6) we obtain that if t solves (8.7) then

$$h'_t(t, \widehat{u}) = 2t - \int_{\mathbb{R}^N} \left(f'_s(x, t\widehat{u})t\widehat{u}^2 + 2mt\widehat{u}^2\right) - \int_{\mathbb{R}^N} f(x, t\widehat{u})\widehat{u} < 0. \tag{8.8}$$

This indicates that for each $\widehat{u} \in \widehat{S}$,

$$\sup_{t\in[0,+\infty)} J\left(t\widehat{u}\right)=J\left(t\left(\widehat{u}\right)\widehat{u}\right)<+\infty. \tag{8.9}$$

For $\forall \widehat{u_0}\in\widehat{S}$, by (8.8) and IFT (implicit function theorem), there exist r, $r_1>0$ and unique $t\left(\widehat{u}\right)\in C^1\left(B\left(\widehat{u}_0,r\right)\cap\widehat{S}, B\left(t_0,r_1\right)\right)$ such that $t\left(\widehat{u}\right)\widehat{u}\in \mathrm{N}$ and $h_t'\left(t\left(\widehat{u}\right),\widehat{u}\right)<0$, where $t_0=t\left(\widehat{u}_0\right)$, $B\left(t_0,r_1\right)=\left(t_0-r_1,t_0+r_1\right)$. Observe that $t\left(\widehat{u}\right)\widehat{u}$ is a diffeomorphism from $B\left(\widehat{u}_0,r\right)\cap\widehat{S}$ to $N_{u_0}=\left\{u=t\left(\widehat{u}\right)\widehat{u}\in\mathrm{N}:\widehat{u}\in B\left(\widehat{u}_0,r\right)\cap\widehat{S}\right\}$, so $\widehat{S}$ is an open set in S_1. It is easy to see that $J\left(t\left(\widehat{u}_n\right)\widehat{u}_n\right)\to+\infty$ as $\widehat{u}_n\to\widehat{u}\in\partial\overline{\widehat{S}}$, where $\overline{\widehat{S}}$ is a closure of $\widehat{S}$ in S_1. Since $\widehat{S}$ is a C^1 manifold, N is also a C^1 manifold. Combining $h_t'\left(t\left(\widehat{u}\right),\widehat{u}\right)<0$ and IFT we know that $\partial\mathrm{N}=\varnothing$. As a submanifold of E with its canonical metric, N inherits a Riemann structure and consequently N is a H–R (Hilbert–Riemann) manifold. □

8.3. *Brezis–Martin theorem and separability of $+P\cap\mathrm{N}$ with $-P\cap\mathrm{N}$*

Define $+P=\{u\in E:u\geq 0\}$ and $-P=\{u\in E:u\leq 0\}$. The following local existence and uniqueness result have a positive effect on proving the invariant property of $\pm P$ under the negative gradient flow of J (see [6], [8] and [28]).

Proposition 8.4. *(Brezis–Martin) Let A be an open subset of a Banach space X, and let $B\subset A$ be closed in A. If $\widetilde{V}:A\to X$ is a locally Lipschitz mapping, then for $\forall u\in B$, $\exists r>0$ and $\eta\left(t,u\right)$, satisfying*

$$\begin{cases}\dot{\eta}\left(t,u\right)=\widetilde{V}\left(\eta\left(t,u\right)\right), & \forall t\in[0,r),\\ \eta\left(0,u\right)=u\in B, & \eta\left(t,u\right)\in B,\end{cases} \tag{8.10}$$

if and only if

$$\lim_{h\searrow 0} h^{-1}d\left(u+h\widetilde{V}\left(u\right),B\right)=0. \tag{8.11}$$

Lemma 8.5. *(Invariant property of $\pm P$) Under the assumptions (f_1), (f_3), (f_6), $\pm P$ are invariant sets under the negative gradient flow $\eta\left(t,u\right)$ of J, i.e., $\eta\left(t,u\right)\in\pm P$, $\forall u\in\pm P$, $t\geq 0$.*

Proof. Let $\eta\left(t,u\right)$ be the negative gradient flow of J given by

$$\begin{cases}\frac{d\eta(t,u)}{dt}=-\nabla J\left(\eta\left(t,u\right)\right),\\ \eta\left(0,u\right)=u.\end{cases} \tag{8.12}$$

By (f_1), (f_3) and Lemma 7.4, the maximal interval of existence of $\eta\left(t,u\right)$ is $[0,+\infty)$. Take $A=E$, $B=+P$, $\widetilde{V}=-\nabla J$, $r=+\infty$ in Proposition 8.4. We now show

$$\lim_{h\searrow 0} h^{-1}d\left(u-h\nabla J\left(u\right),+P\right)=0. \tag{8.13}$$

Notice that

$$\inf_{v\in+P}\left\|u-h\nabla J\left(u\right)-v\right\|_m\leq\left\|\left(u-h\nabla J\left(u\right)\right)^-\right\|_m, \tag{8.14}$$

invoking Lemma 8.2 and (f_6),

$$A_m^{-1} g_m(x,u) \geq 0, \forall u \in +P, \tag{8.15}$$

and hence

$$\begin{aligned} &\lim_{h \searrow 0} h^{-1} d\left(u - h\nabla J(u), +P\right) \\ &\leq \lim_{h \searrow 0} \left\| h^{-1}\left(u - h\nabla J(u)\right)^{-} \right\|_m \\ &= \left\| \lim_{h \searrow 0} \left[h^{-1}u - u + A_m^{-1} g_m(x,u) \right]^{-} \right\|_m = 0. \end{aligned} \tag{8.16}$$

Proceeding along the same lines, we have

$$\lim_{h \searrow 0} h^{-1} d\left(u - h\nabla J(u), -P\right) = 0. \tag{8.17}$$

The assertion follows. □

Recall that if M is a H–R manifold in Hilbert space X and $\Omega(u,v)$ denotes the family of all piecewise smooth curves connected u, v in M, then we define

$$d_M(u,v) = \inf_{\sigma \in \Omega(u,v)} \int_a^b \left\| \dot{\sigma}(\tau) \right\| d\tau, \tag{8.18}$$

where $\sigma(a) = u$, $\sigma(b) = v$, $\forall \sigma \in \Omega(u,v)$.

Remark 8.6. Palais [31] (see also [8]) showed that d_M is a metric on M and the reduced topology is equivalent to the topology on the manifold.

Theorem 8.7. *(Separation theorem) Under the hypotheses (f_3)–(f_6), there exists a $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$, then*

(i) $\left((+P)^\delta \cap \mathrm{N}\right) \cap \left((-P)^\delta \cap \mathrm{N}\right) = \varnothing$;

(ii) $\eta(t,u) \in \left((\pm P)^\delta\right)^\circ$ *as* $u \in (\pm P)^\delta$, $\forall t \in (0, +\infty)$.

Proof. First it is easy to see that $(+P \cap \mathrm{N}) \cap (-P \cap \mathrm{N}) = \varnothing$. Otherwise, $\exists u \in (+P \cap \mathrm{N}) \cap (-P \cap \mathrm{N})$, and this implies that $u = 0$, contradicting $u \in \mathrm{N}$. By way of negation, $\exists \delta_n \to 0$, $u_n \in \mathrm{N}$, s.t. $u_n \in \left((+P)^{\delta_n} \cap \mathrm{N}\right) \cap \left((-P)^{\delta_n} \cap \mathrm{N}\right)$.

Note that

$$\inf_{w \in +P} \|u - w\|_{L^2} = \left\|u^-\right\|_{L^2} \tag{8.19}$$

and

$$\inf_{w \in +P} \|u_n - w\|_{L^2} \leq C \inf_{w \in +P} \|u_n - w\|_m \leq C\delta_n, \tag{8.20}$$

thereby,

$$u_n^- \to 0 \text{ in } L^2\left(\mathbb{R}^N\right). \tag{8.21}$$

An analogous argument yields

$$u_n^+ \to 0 \text{ in } L^2\left(\mathbb{R}^N\right). \tag{8.22}$$

This together with (8.21) shows

$$u_n \to 0 \text{ in } L^2\left(\mathbb{R}^N\right). \tag{8.23}$$

Notice that

$$\|u_n\|_m^2 = \int_{\mathbb{R}^N} g_m(x, u_n) u_n \le C_1 \|u_n\|^2. \tag{8.24}$$

Hence by (8.23),

$$\|u_n\|_m \to 0. \tag{8.25}$$

We now claim that

$$\inf_{u \in \mathrm{N}} \|u\|_m > 0. \tag{8.26}$$

In view of the hypotheses (f_3) and (f_4), for fixed $p \in (2, 2^*)$, $\widetilde{g}(x, s) = o\left(|s|^{p-1}\right)$ as $s \to \infty$, uniformly on $x \in \mathbb{R}^N$, and therefore for $\forall \varepsilon > 0$, $\exists C(\varepsilon) > 0$, s.t.

$$|\widetilde{g}(x, s)| \le \varepsilon |s| + C(\varepsilon) |s|^{p-1}, \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}. \tag{8.27}$$

Suppose to the contrary that $\exists u_n^* \in \mathrm{N}$, $\left\|u_n^*\right\|_m \to 0$. Set $u_n^* = t_n \widehat{u}_n$, $t_n > 0$, $t_n \to 0$, $\|\widehat{u}_n\|_m = 1$. By (f_5), we can pick $\varepsilon > 0$ small sufficiently such that $\Lambda(a_0, b_0) + \varepsilon < \lambda_1$. Consequently,

$$\begin{aligned}
1 &= \int_{\mathbb{R}^N} |\nabla \widehat{u}_n|^2 + (V + m) \widehat{u}_n^2 \\
&\le \int_{\mathbb{R}^N} \left(a_0 \left|\widehat{u_n^-}\right|^2 + b_0 \left|\widehat{u_n^+}\right|^2\right) + (\varepsilon + m) \int_{\mathbb{R}^N} |\widehat{u}_n|^2 + t_n^{p-2} C(\varepsilon) \int_{\mathbb{R}^N} |\widehat{u}_n|^p \\
&\le (\Lambda(a_0, b_0) + \varepsilon + m) \int_{\mathbb{R}^N} |\widehat{u}_n|^2 + t_n^{p-2} C(\varepsilon) \int_{\mathbb{R}^N} |\widehat{u}_n|^p \\
&< \frac{\Lambda(a_0, b_0) + \varepsilon + m}{\lambda_1 + m} + t_n^{p-2} C(\varepsilon) \int_{\mathbb{R}^N} |\widehat{u}_n|^p,
\end{aligned} \tag{8.28}$$

yielding a contradiction

$$1-\frac{\Lambda\left(a_0,b_0\right)+\varepsilon+m}{\lambda_1+m}<t_n^{p-2}C\left(\varepsilon\right)\int\limits_{\mathbb{R}^N}\left|\widehat{u}_n\right|^p\to 0. \tag{8.29}$$

The claim is thus proved, conflicting with (8.25). That's the precise statement (i).

(8.27) shows that $\exists\rho>0$, for fixed $q\in(2,2^*)$, $\exists K>0$, s.t.

$$\left|g_m\left(x,s\right)\right|\leq\left(\lambda_1+m-\rho\right)\left|s\right|+K\left|s\right|^{q-1},\forall\left(x,s\right)\in\mathbb{R}^N\times\mathbb{R}, \tag{8.30}$$

and using (f_6) we get $g_m\left(x,s\right)s>0$, $\forall s\neq 0$, a.e. on $x\in\mathbb{R}^N$.

Notice that for any $u\in(+P)^\delta$ and $v=A_m^{-1}g_m\left(x,u\right)$,

$$\begin{aligned}
&\inf_{w\in+P}\|v-w\|_m\cdot\left\|v^-\right\|_m\\
&\leq\left\|v^-\right\|_m^2=\left\langle v,v^-\right\rangle_m\\
&=\int\limits_{\mathbb{R}^N}g_m\left(x,u\right)v^-\leq\int\limits_{\mathbb{R}^N}g_m\left(x,u^-\right)v^-\\
&\leq\left(\lambda_1+m-\rho\right)\left\|u^-\right\|_{L^2}\cdot\left\|v^-\right\|_{L^2}+K\left\|u^-\right\|_{L^q}^{q-1}\cdot\left\|v^-\right\|_{L^q}\\
&=\left(\lambda_1+m-\rho\right)\inf_{w\in+P}\|u-w\|_{L^2}\cdot\left\|v^-\right\|_{L^2}\\
&\quad+K\inf_{w\in+P}\|u-w\|_{L^q}^{q-1}\cdot\left\|v^-\right\|_{L^q}\\
&\leq\frac{\lambda_1+m-\rho}{\lambda_1+m}\inf_{w\in+P}\|u-w\|_m\cdot\left\|v^-\right\|_m\\
&\quad+\widetilde{K}\inf_{w\in+P}\|u-w\|_m^{q-1}\cdot\left\|v^-\right\|_m\\
&\leq\left(\frac{\lambda_1+m-\rho}{\lambda_1+m}+\widetilde{K}\delta^{q-2}\right)\cdot\inf_{w\in+P}\|u-w\|_m\cdot\left\|v^-\right\|_m.
\end{aligned} \tag{8.31}$$

In virtue of (8.31), it follows that for $\delta>0$ sufficiently small, $\exists\beta\in(0,1)$, s.t.

$$\inf_{w\in+P}\|v-w\|_m\leq\beta\cdot\inf_{w\in+P}\|u-w\|_m\leq\beta\delta, \tag{8.32}$$

i.e.,

$$v\subset(+P)^{\beta\delta}\subset\left((+P)^\delta\right)^\circ\text{ as }u\in(+P)^\delta. \tag{8.33}$$

Similarly we have

$$v\subset(-P)^{\beta\delta}\subset\left((-P)^\delta\right)^\circ\text{ as }u\in(-P)^\delta. \tag{8.34}$$

Hence, $\exists w_u,w_v\in+P$,

$$\|u - w_u\|_m = \inf_{w \in +P} \|u - w\|_m \le \delta, \tag{8.35}$$

$$\|v - w_v\|_m = \inf_{w \in +P} \|v - w\|_m \le \delta, \tag{8.36}$$

therefore,

$$\begin{aligned} &\inf_{w \in +P} \left\| u - hJ'(u) - w \right\|_m \\ &= \inf_{v \in +P} \|(1-h)u + hv - w\|_m \\ &\le (1-h)\|u - w_u\|_m + h\|v - w_v\|_m \le \delta, \end{aligned} \tag{8.37}$$

yielding

$$\inf_{w \in (+P)^\delta} \left\| u - hJ'(u) - w \right\|_m = 0. \tag{8.38}$$

Likewise, we have

$$\inf_{w \in (-P)^\delta} \left\| u - hJ'(u) - w \right\|_m = 0. \tag{8.39}$$

By Proposition 8.4 we consequently verified (ii). The proof of Theorem 8.7 is complete. □

Lemma 8.8. *Under the assumptions* (V_2)–(V_4), (f_1), (f_3)–(f_6), *if* $(a,b) \notin \Sigma(A)$, *then* J *has at least two nontrivial critical points* $u_1 \in +P \cap \mathrm{N}$, $u_2 \in -P \cap \mathrm{N}$. *Moreover, if* $K_J \cap (+P)^\delta = \{0, u_1\}$, $K_J \cap (-P)^\delta = \{0, u_2\}$, *then* $C_q(J, u_i) \cong \delta_{q1} G$, $i = 1, 2$, *where* $C_q(J, u_i)$ *denotes the* q*-th critical group, with coefficient group G of* J *at* u_i.

Proof. For $\forall \varphi \in \left((+P)^\delta\right)^\circ \cap \mathrm{N}$, define

$$\Gamma_{+,\varphi,\delta} = \left\{ \begin{array}{c} h(t) : h \in C\left([0,1], \left((+P)^\delta\right)^\circ\right), \text{with } h(0) = 0, h(1) = c\widehat{\varphi}, \\ \text{where } \widehat{\varphi} = \frac{\varphi}{\|\varphi\|}, c > t(\widehat{\varphi}) > 0 \text{ such that } J(c\widehat{\varphi}) < 0, \end{array} \right\}$$

and for $\forall \varphi \in \left((-P)^\delta\right)^\circ \cap \mathrm{N}$, denote

$$\Gamma_{-,\varphi,\delta} = \left\{ \begin{array}{c} h(t) : h \in C\left([0,1], \left((-P)^\delta\right)^\circ\right), \text{with } h(0) = 0, h(1) = c\widehat{\varphi}, \\ \text{where } \widehat{\varphi} = \frac{\varphi}{\|\varphi\|}, c > t(\widehat{\varphi}) > 0 \text{ such that } J(c\widehat{\varphi}) < 0. \end{array} \right\}$$

By (f_5), 0 is a local minimizer of J and there exist $\rho, r > 0$ such that

$$\inf_{u \in \partial B(0,r)} J(u) \ge \rho. \tag{8.40}$$

In view of Theorem 8.7, a standard argument shows that

$$C_{1,\varphi,\delta} = \inf_{h(t) \in \Gamma_{+,\varphi,\delta}} \sup_{u \in h(t)} J(u) \ge \rho \tag{8.41}$$

is a critical value of J and the corresponding critical point $u_{1,\varphi,\delta} \in \left((+P)^{\delta}\right)^{\circ}$. Since δ is arbitrary, there exists $u_{1,\varphi} \in +P \cap \mathrm{N}$. This follows directly from step three of the proof of Theorem 3.4, and thus $u_{1,\varphi,\delta}$ has a convergent subsequence as $\delta \to 0$.

Set

$$C_1 = \inf_{\varphi \in \left((+P)^{\delta}\right)^{\circ} \cap \mathrm{N}} J\left(u_{1,\varphi}\right). \tag{8.42}$$

Then by Theorem 3.4 there exists $u_1 \in +P \cap \mathrm{N}$, $J(u_1) = C_1 > 0$, $J'(u_1) = 0$.

Similarly, for $\forall \varphi \in \left((-P)^{\delta}\right)^{\circ} \cap \mathrm{N}$,

$$C_{2,\varphi,\delta} = \inf_{h(t) \in \Gamma_{-,\varphi,\delta}} \sup_{u \in h(t)} J(u) \tag{8.43}$$

is a critical value of J and $\exists u_{2,\varphi} \in -P \cap K_J \setminus \{0\}$. Take

$$C_2 = \inf_{\varphi \in \left((-P)^{\delta}\right)^{\circ} \cap \mathrm{N}} J\left(u_{2,\varphi}\right), \tag{8.44}$$

and thus $\exists u_2 \in -P \cap \mathrm{N}$, $J(u_2) = C_2 > 0$, $J'(u_2) = 0$.

We now claim that if u_1 is the unique critical point of J in $+P$, then

(i) $\left((+P)^{\delta}\right)^{\circ} \cap J^{C_1} \setminus \{u_1\}$ is not path-connected;

(ii) $\left((+P)^{\delta}\right)^{\circ} \cap J^{C_1}$ is path-connected.

Otherwise, if $\left((+P)^{\delta}\right)^{\circ} \cap J^{C_1} \setminus \{u_1\}$ is path-connected, then due to the facts that $\widehat{u}_1 \in \widehat{S}$ and $\widehat{S}$ is an open set in S_1, we can take v_1, v_2 such that $\widehat{v}_1, \widehat{v}_2 \in \widehat{S}$, $t(\widehat{v}_1) \geq \|v_1\|_m$, $t(\widehat{v}_2) \leq \|v_2\|_m$, and there exist $r_1 > 0$, and a path $r(s) \in \left((+P)^{\delta}\right)^{\circ}$ for $\forall s \in [0,1]$, with $r(0) = v_1$, $r(1) = v_2$, $\inf_{s \in [0,1]} \|u_1 - r(s)\|_m > r_1$, $\sup_{s \in [0,1]} J(r(s)) \leq C_1$.

Let

$$\widetilde{r}(\tau) = \begin{cases} 3\tau v_1, & \tau \in \left[0, \frac{1}{3}\right], \\ r\left(3\left(\tau - \frac{1}{3}\right)\right), & \tau \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 3(1-\tau) v_2 + 3\left(\tau - \frac{2}{3}\right) c\widehat{v}_2, & \tau \in \left[\frac{2}{3}, 1\right], \end{cases}$$

where $\widehat{v}_i = \frac{v_i}{\|v_i\|_m}$, $c > t(\widehat{v}_2)$ such that $J(c\widehat{v}_2) < 0$. It is evident that $\widetilde{r}(\tau) \in \Gamma_{+,t(\widehat{v}_2)\widehat{v}_2,\delta}$, $\sup_{\tau \in [0,1]} J(\widetilde{r}(\tau)) \leq C_1$. Employing quantitative deformation lemma (see [41]), we can find $\varepsilon > 0$, a flow $\xi(t,u) : [0,1] \times E \to E$, s.t.

$$\|\xi(t,u) - u\|_m < \frac{r_1}{2}, \forall u \in \left((+P)^{\delta}\right)^{\circ}, \tag{8.45}$$

$$\xi(1, \widetilde{r}(\tau)) \in \Gamma_{+,t(\widehat{v}_2)\widehat{v}_2,\delta}, \tau \in [0,1], \tag{8.46}$$

$$\xi(1, \widetilde{r}(\tau)) \subset J^{C_1 - \varepsilon} \cap \left((+P)^{\delta}\right)^{\circ}, \tau \in [0,1]. \tag{8.47}$$

Combining (8.45), (8.46) and (8.47) we derive an expected contradiction

$$C_1 - \varepsilon > \sup_{\tau\in[0,1]} J\left(\xi\left(1, \widetilde{r}(\tau)\right)\right) \geq C_{1,v_2,\delta} \geq C_1, \tag{8.48}$$

concluding the proof of (i).

Next, for any $v_1, v_2 \in \left((+P)^\delta\right)^\circ \cap J^{C_1}$, we intend to indicate that there exists a path r connecting v_1 and v_2, s.t.

$$r(s) \in \left((+P)^\delta\right)^\circ, \forall s \in [0,1], \tag{8.49}$$

$$\sup_{s\in[0,1]} J(r(s)) \leq C_1. \tag{8.50}$$

As a matter of fact, we can define

$$r_1(s) = \begin{cases} 2\left(\frac{1}{2} - s\right) v_1, & s \in \left[0, \frac{1}{2}\right], \\ 2\left(s - \frac{1}{2}\right) v_2, & s \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

By second deformation lemma there exists $\zeta(t,u) : [0,1] \times (+P)^\delta \to J^{C_1} \cap (+P)^\delta$ such that $J^{C_1} \cap (+P)^\delta$ is a strong deformation retract of $(+P)^\delta$. Take $r(s) = \zeta(1, r_1(s))$. It is easy to check that $r(0) = v_1$, $r(1) = v_2$, and (8.49), (8.50) follow.

Consider the following exact sequence

$$\begin{aligned} \cdots \to\ & H_1\left(J^{C_1} \cap \left((+P)^\delta\right)^\circ, J^{C_1} \cap \left((+P)^\delta\right)^\circ \setminus \{u_1\}\right) \\ & \xrightarrow{\partial} H_0\left(J^{C_1} \cap \left((+P)^\delta\right)^\circ \setminus \{u_1\}\right) \xrightarrow{i_*} H_0\left(J^{C_1} \cap \left((+P)^\delta\right)^\circ\right) \\ & \xrightarrow{j_*} H_0\left(J^{C_1} \cap \left((+P)^\delta\right)^\circ, J^{C_1} \cap \left((+P)^\delta\right)^\circ \setminus \{u_1\}\right) \\ \to\ & \cdots, \end{aligned} \tag{8.51}$$

where

$$i : J^{C_1} \cap \left((+P)^\delta\right)^\circ \setminus \{u_1\} \to J^{C_1} \cap \left((+P)^\delta\right)^\circ$$

and

$$j : J^{C_1} \cap \left((+P)^\delta\right)^\circ \to \left(J^{C_1} \cap \left((+P)^\delta\right)^\circ, J^{C_1} \cap \left((+P)^\delta\right)^\circ \setminus \{u_1\}\right)$$

are the inclusions, and ∂ is the boundary operator.

Set $X = J^{C_1} \cap \left((+P)^\delta\right)^\circ$, $Y = J^{C_1} \cap \left((+P)^\delta\right)^\circ \setminus \{u_1\}$. Notice that $\exists r > 0$ small suitably, s.t. $B_m(0,r) \subset Y^\circ$. By using the excision property

$$C_0(J, u_1) \cong H_0\left(X \backslash B_m(0,r), Y \backslash B_m(0,r)\right) \cong H_0(X, Y) \cong 0. \tag{8.52}$$

As

$$H_0\left(J^{C_1} \cap \left((+P)^\delta\right)^\circ\right) \cong G, \tag{8.53}$$

and

$$\operatorname{rank} H_0\left(J^{C_1}\cap\left((+P)^{\delta}\right)^{\circ}\setminus\{u_1\}\right)\geq 2, \tag{8.54}$$

(8.51) yields

$$C_1(J,u_1)\cong H_1(X,Y)\neq 0. \tag{8.55}$$

Along the same lines, if u_2 is the unique critical point of J in $-P\setminus\{0\}$, then we have

$$C_1(J,u_2)\neq 0, \tag{8.56}$$

and in terms of Lemma 7.8, we conclude the proof. □

8.4. The proof of Theorem 1.1

So far we have got a positive solution u_1 and a negative solution u_2 in terms of Lemma 8.8. To find a sign-changing solution u_3, it is necessary for us to use the method based on the work of [26] and [25]. Let $\eta(t,u)$ be the negative gradient flow given by (8.12) and consider

$$\begin{aligned} O_{\pm} &= \left\{u\in E:\eta(t,u)\in(\pm P)^{\delta} \text{ for some } t\in(0,+\infty)\ \right\};\\ O &= O_+\cap O_-, \end{aligned}$$

then O, O_+, O_- are the open neighborhoods of 0, $+P$, $-P$ respectively.

In view of [26], (f_4) yields

$$\partial O\cap P\neq\varnothing, \partial O\cap(-P)\neq\varnothing, \partial O\setminus(O_+\cup O_-)\neq\varnothing. \tag{8.57}$$

Hence one can find a critical point u_3 in $\partial O\setminus(O_+\cup O_-)$, and of course it is a sign-changing solution of (1.1).

Without loss of generality, suppose (a,b) is below $C_{k+1}^{(1)}$. Assume that (1.1) has only three nontrivial solutions u_1, u_2 and u_3, we will be devoted to showing

$$J(u_3)>\Lambda(J(u_1),J(u_2)). \tag{8.58}$$

Suppose to the contrary. Clearly, $\inf\limits_{u\in\mathrm{N}} J(u)>0\Rightarrow \inf\limits_{u\in K_J\setminus\{0\}} J(u)>0$. On account of the hypotheses of Theorem 1.1, invoking Theorem 6.2, $C_q(J,\infty)\cong\delta_{qd_k}G$. Therefore, by the following exact sequence

$$\cdots\to H_{q+1}(E)\xrightarrow{j_*}H_{q+1}\left(E,J^{-\varepsilon}\right)\xrightarrow{\partial}H_q\left(J^{-\varepsilon}\right)\xrightarrow{i_*}H_q(E)\to\cdots, \tag{8.59}$$

we get

$$H_q\left(J^{-\varepsilon}\right)\cong H_{q+1}\left(E,J^{-\varepsilon}\right)=C_{q+1}(J,\infty)\cong\delta_{q+1d_k}G, q\geq 1. \tag{8.60}$$

Set $C_1=J(u_1)$, $C_2=J(u_2)$. Take $\varepsilon>0$ suitably small, s.t. $K_J\cap J^{\varepsilon}=\{0\}$. As $C_q(J,0)\cong\delta_{q0}G$, it follows that

$$H_q\left(J^{\varepsilon}\right)\cong H_q\left(J^{-\varepsilon}\right)\cong \delta_{q+1d_k}G, q\geq 1. \tag{8.61}$$

Without loss of generality, $C_1 \leq C_2$. We are merely to deal with the case $C_1 < C_2$. Take four cases into account:

Case 1: $C_3 < C_1$. For above $\varepsilon > 0$, s.t., $3\varepsilon < C_3+\varepsilon < C_1$, according to Lemma 8.8 and Morse inequalities

$$\begin{aligned} 0 &= \sum_{i=1}^{2} \operatorname{rank} C_q\left(J, u_i\right) = M_q\left(C_3+\varepsilon, C_2+\varepsilon\right) \\ &\geq \beta_q\left(C_3+\varepsilon, C_2+\varepsilon\right) = \operatorname{rank} H_q\left(J^{C_2+\varepsilon}, J^{C_3+\varepsilon}\right) \end{aligned} \tag{8.62}$$

for $q\geq 2$, alluding to

$$H_q\left(J^{C_2+\varepsilon}, J^{C_3+\varepsilon}\right)\cong 0, q\geq 2. \tag{8.63}$$

Since $H_q\left(J^{C_2+\varepsilon}\right)\cong H_q\left(E\right)$, we have

$$H_q\left(J^{C_3+\varepsilon}\right)\cong H_{q+1}\left(J^{C_2+\varepsilon}, J^{C_3+\varepsilon}\right)\cong 0, q\geq 1. \tag{8.64}$$

An argument analogous to [3] and [1] shows that

$$n\left(u_i\right)\leq m\left(u_i\right) \tag{8.65}$$

holds for $\Omega=\mathbb{R}^N$, where $n\left(u_i\right)$ stands for the number of nodal domains of u_i, and then $m\left(u_i\right)\geq 1, i=1,2$.

Due to $H_0\left(E, J^{-\varepsilon}\right)\cong 0$, by the short exact sequence below

$$\cdots\to H_1\left(E, J^{-\varepsilon}\right)\overset{\partial}{\to} H_0\left(J^{-\varepsilon}\right)\overset{i_*}{\to} H_0\left(E\right)\overset{j_*}{\to} H_0\left(E, J^{-\varepsilon}\right)\to\cdots, \tag{8.66}$$

we obtain

$$H_0\left(J^{-\varepsilon}\right)\cong H_0\left(E\right)\cong G. \tag{8.67}$$

Also observe that

$$\begin{aligned} \cdots\to H_1\left(J^{\varepsilon}, J^{-\varepsilon}\right)&\overset{\partial}{\to} H_0\left(J^{-\varepsilon}\right)\overset{i_*}{\to} H_0\left(J^{\varepsilon}\right) \\ &\overset{j_*}{\to} H_0\left(J^{\varepsilon}, J^{-\varepsilon}\right)\overset{\partial}{\to} H_{-1}\left(J^{-\varepsilon}\right) \\ &\to\cdots \end{aligned} \tag{8.68}$$

which via (8.67) implies

$$\operatorname{rank} H_0\left(J^{\varepsilon}\right)=\operatorname{rank} H_0\left(J^{-\varepsilon}\right)+1=2, \tag{8.69}$$

Note that $m(u_3) \geq n(u_3) \geq 2$, so $C_0(J, u_3) \cong C_1(J, u_3) \cong 0$. In virtue of $H_q\left(J^{C_3-\varepsilon}\right) \cong H_q(J^{\varepsilon})$, invoking the exact sequence

$$\begin{aligned} \cdots \to H_1\left(J^{C_3+\varepsilon}, J^{C_3-\varepsilon}\right) &\xrightarrow{\partial} H_0\left(J^{C_3-\varepsilon}\right) \\ &\xrightarrow{i_*} H_0\left(J^{C_3+\varepsilon}\right) \xrightarrow{j_*} H_0\left(J^{C_3+\varepsilon}, J^{C_3-\varepsilon}\right) \cdots, \end{aligned} \tag{8.70}$$

one deduces that

$$H_0\left(J^{C_3+\varepsilon}\right) \cong H_0\left(J^{C_3-\varepsilon}\right) \cong G \oplus G. \tag{8.71}$$

Evidently, $M_0(C_3+\varepsilon, C_2+\varepsilon) = \beta_0(C_3+\varepsilon, C_2+\varepsilon) = 0$. Employing Morse equality

$$\sum_{q=0}^{+\infty}(-1)^q M_q(C_3+\varepsilon, C_2+\varepsilon) = \sum_{q=0}^{+\infty}(-1)^q \beta_q(C_3+\varepsilon, C_2+\varepsilon) \tag{8.72}$$

indicates that

$$\begin{aligned} 2 = \sum_{i=1}^{2} \operatorname{rank} C_1(J, u_i) &= M_1(C_3+\varepsilon, C_2+\varepsilon) \\ &= \beta_1(C_3+\varepsilon, C_2+\varepsilon) = \operatorname{rank} H_1\left(J^{C_2+\varepsilon}, J^{C_3+\varepsilon}\right), \end{aligned} \tag{8.73}$$

yielding $\operatorname{rank} H_0\left(J^{C_3+\varepsilon}\right) = 3$, violating (8.71).

Case 2: $C_3 = C_1$. Consider the exact sequence

$$\begin{aligned} &\cdots \xrightarrow{j_*} H_2\left(J^{C_2+\varepsilon}, J^{C_2-\varepsilon}\right) \xrightarrow{\partial} H_1\left(J^{C_2-\varepsilon}\right) \xrightarrow{i_*} H_1\left(J^{C_2+\varepsilon}\right) \\ &\xrightarrow{j_*} H_1\left(J^{C_2+\varepsilon}, J^{C_2-\varepsilon}\right) \xrightarrow{\partial} H_0\left(J^{C_2-\varepsilon}\right) \xrightarrow{i_*} H_0\left(J^{C_2+\varepsilon}\right) \\ &\xrightarrow{j_*} H_0\left(J^{C_2+\varepsilon}, J^{C_2-\varepsilon}\right) \cdots \end{aligned} \tag{8.74}$$

and observe that

$$H_2\left(J^{C_2+\varepsilon}, J^{C_2-\varepsilon}\right) \cong H_1\left(J^{C_2+\varepsilon}\right) \cong H_0\left(J^{C_2+\varepsilon}, J^{C_2-\varepsilon}\right) \cong 0, \tag{8.75}$$

we derive $H_1\left(J^{C_2-\varepsilon}\right) \cong 0$ and

$$\begin{aligned} H_0\left(J^{C_2-\varepsilon}\right) &\cong H_1\left(J^{C_2+\varepsilon}, J^{C_2-\varepsilon}\right) \oplus H_0\left(J^{C_2+\varepsilon}\right) \\ &\cong G \oplus G. \end{aligned} \tag{8.76}$$

Thanks to $H_0\left(J^{C_2-\varepsilon}, J^{\varepsilon}\right) \cong C_0(J, u_1) \oplus C_0(J, u_3)$, based on the exact sequence

$$\begin{aligned}\cdots \to H_1\left(J^{C_2-\varepsilon}\right) &\xrightarrow{j_*} H_1\left(J^{C_2-\varepsilon}, J^{\varepsilon}\right) \xrightarrow{\partial} H_0\left(J^{\varepsilon}\right)\\ &\xrightarrow{i_*} H_0\left(J^{C_2-\varepsilon}\right) \xrightarrow{j_*} H_0\left(J^{C_2-\varepsilon}, J^{\varepsilon}\right)\cdots,\end{aligned} \tag{8.77}$$

we get

$$\begin{aligned}H_0\left(J^{\varepsilon}\right) &\cong H_1\left(J^{C_2-\varepsilon}, J^{\varepsilon}\right) \oplus H_0\left(J^{C_2-\varepsilon}\right)\\ &\cong G \oplus G \oplus G,\end{aligned} \tag{8.78}$$

contradicting (8.69).

Case 3: $C_2 > C_3 > C_1$. Notice that $H_1(E) \cong H_0\left(E, J^{C_3+\varepsilon}\right) \cong 0$, by the exact sequence

$$\begin{aligned}\cdots &\xrightarrow{i_*} H_1(E) \xrightarrow{j_*} H_1\left(E, J^{C_3+\varepsilon}\right)\\ &\xrightarrow{\partial} H_0\left(J^{C_3+\varepsilon}\right) \xrightarrow{i_*} H_0(E) \xrightarrow{j_*} H_0\left(E, J^{C_3+\varepsilon}\right)\\ &\to \cdots\end{aligned} \tag{8.79}$$

we have

$$\begin{aligned}H_0\left(J^{C_3+\varepsilon}\right) &\cong H_1\left(E, J^{C_3+\varepsilon}\right) \oplus H_0(E)\\ &\cong H_1\left(J^{C_2+\varepsilon}, J^{C_2-\varepsilon}\right) \oplus H_0(E)\\ &\cong G \oplus G.\end{aligned} \tag{8.80}$$

Again observe $C_0(J, u_3) \cong C_1(J, u_3) \cong 0$, invoking the exact sequence

$$\begin{aligned}\cdots &\xrightarrow{j_*} H_1\left(J^{C_3+\varepsilon}, J^{C_3-\varepsilon}\right) \xrightarrow{\partial} H_0\left(J^{C_3-\varepsilon}\right)\\ &\xrightarrow{i_*} H_0\left(J^{C_3+\varepsilon}\right) \xrightarrow{j_*} H_0\left(J^{C_3+\varepsilon}, J^{C_3-\varepsilon}\right)\cdots\end{aligned} \tag{8.81}$$

shows

$$H_0\left(J^{C_3-\varepsilon}\right) \cong H_0\left(J^{C_3+\varepsilon}\right). \tag{8.82}$$

As $H_1\left(J^{-\varepsilon}\right) \cong H_{-1}\left(J^{-\varepsilon}\right) \cong 0$, based on the following exact sequence

$$\begin{aligned}\cdots \to H_1\left(J^{-\varepsilon}\right) &\xrightarrow{i_*} H_1\left(J^{C_3-\varepsilon}\right) \xrightarrow{j_*} H_1\left(J^{C_3-\varepsilon}, J^{-\varepsilon}\right)\\ &\xrightarrow{\partial} H_0\left(J^{-\varepsilon}\right) \xrightarrow{i_*} H_0\left(J^{C_3-\varepsilon}\right) \xrightarrow{j_*} H_0\left(J^{C_3-\varepsilon}, J^{-\varepsilon}\right)\\ &\to H_{-1}\left(J^{-\varepsilon}\right)\cdots,\end{aligned} \tag{8.83}$$

one deduces

$$\begin{aligned}
&\operatorname{rank} H_1\left(J^{C_3-\varepsilon}\right)-\operatorname{rank} H_1\left(J^{C_3-\varepsilon}, J^{-\varepsilon}\right)+\operatorname{rank} H_0\left(J^{-\varepsilon}\right) \\
&-\operatorname{rank} H_0\left(J^{C_3-\varepsilon}\right)+\operatorname{rank} H_0\left(J^{C_3-\varepsilon}, J^{-\varepsilon}\right) \\
=\;& 0.
\end{aligned} \tag{8.84}$$

Thanks to $M_q(-\varepsilon, C_3-\varepsilon)=\beta_q(-\varepsilon, C_3-\varepsilon)=0$ for $q \geq 2$, an argument analogous to Case 1 indicates

$$\operatorname{rank} H_1\left(J^{C_3-\varepsilon}, J^{-\varepsilon}\right)=\operatorname{rank} H_0\left(J^{C_3-\varepsilon}, J^{-\varepsilon}\right). \tag{8.85}$$

Combining (8.84) with (8.85), we obtain

$$\operatorname{rank} H_0\left(J^{C_3-\varepsilon}\right)=\operatorname{rank} H_1\left(J^{C_3-\varepsilon}\right)+\operatorname{rank} H_0\left(J^{-\varepsilon}\right), \tag{8.86}$$

alluding to

$$H_1\left(J^{C_3-\varepsilon}\right) \cong G. \tag{8.87}$$

As $C_2(J, u_2) \cong H_2\left(J^{C_2+\varepsilon}, J^{C_2-\varepsilon}\right) \cong H_2\left(E, J^{C_3+\varepsilon}\right) \cong 0$, resorting to the short exact sequence

$$\cdots \rightarrow H_2\left(E, J^{C_3+\varepsilon}\right) \xrightarrow{\partial} H_1\left(J^{C_3+\varepsilon}\right) \xrightarrow{i_*} H_1(E) \rightarrow \cdots \tag{8.88}$$

we infer that

$$H_1\left(J^{C_3+\varepsilon}\right) \cong 0. \tag{8.89}$$

Combining (8.87) with (8.88), the following exact sequence

$$\begin{aligned}
\cdots \rightarrow\; & H_2\left(J^{C_3+\varepsilon}, J^{C_3-\varepsilon}\right) \xrightarrow{\partial} H_1\left(J^{C_3-\varepsilon}\right) \\
& \xrightarrow{i_*} H_1\left(J^{C_3+\varepsilon}\right) \xrightarrow{j_*} H_1\left(J^{C_3+\varepsilon}, J^{C_3-\varepsilon}\right) \\
& \rightarrow \cdots
\end{aligned} \tag{8.90}$$

shows

$$C_2(J, u_3) \cong H_2\left(J^{C_3+\varepsilon}, J^{C_3-\varepsilon}\right) \neq 0 \tag{8.91}$$

in terms of $C_1(J, u_3) \cong 0$. Since $m(u_3) \geq n(u_3) \geq 2$, we have $C_q(J, u_3) \cong \delta_{q2} G$. However, according to the hypothesis, $C_q(J, \infty) \cong \delta_{q d_k} G$, $d_k \geq 3$, so there exists a solution u^* of (1.1) with $C_{d_k}(J, u^*) \neq 0$. A paradox!

Case 4: $C_3=C_2>C_1$. Note that $H_q\left(E, J^{C_3-\varepsilon}\right) \cong H_q\left(J^{C_3+\varepsilon}, J^{C_3-\varepsilon}\right)$, in view of the exact sequence

$$\begin{aligned} \cdots \to H_1(E) &\xrightarrow{j_*} H_1\left(E, J^{C_3-\varepsilon}\right) \xrightarrow{\partial} H_0\left(J^{C_3-\varepsilon}\right) \\ &\xrightarrow{i_*} H_0(E) \xrightarrow{j_*} H_0\left(E, J^{C_3-\varepsilon}\right) \\ &\to \cdots \end{aligned} \tag{8.92}$$

we get

$$\begin{aligned} H_0\left(J^{C_3-\varepsilon}\right) &\cong H_1\left(J^{C_3+\varepsilon}, J^{C_3-\varepsilon}\right) \oplus G \\ &\cong C_1(J, u_2) \oplus C_1(J, u_3) \oplus G \\ &\cong G \oplus G. \end{aligned} \tag{8.93}$$

A standard argument yields

$$\operatorname{rank} H_1\left(J^{C_2-\varepsilon}, J^{-\varepsilon}\right) = \operatorname{rank} H_0\left(J^{C_2-\varepsilon}, J^{-\varepsilon}\right). \tag{8.94}$$

Consider the exact sequence

$$\begin{aligned} \cdots \to H_1\left(J^{-\varepsilon}\right) &\xrightarrow{i_*} H_1\left(J^{C_2-\varepsilon}\right) \xrightarrow{j_*} H_1\left(J^{C_2-\varepsilon}, J^{-\varepsilon}\right) \\ &\xrightarrow{\partial} H_0\left(J^{-\varepsilon}\right) \xrightarrow{i_*} H_0\left(J^{C_2-\varepsilon}\right) \xrightarrow{j_*} H_0\left(J^{C_2-\varepsilon}, J^{-\varepsilon}\right) \\ \to H_{-1}\left(J^{-\varepsilon}\right) &\to \cdots \end{aligned} \tag{8.95}$$

and observe that $H_1\left(J^{-\varepsilon}\right) \cong H_{-1}\left(J^{-\varepsilon}\right) \cong 0$, hence

$$\begin{aligned} &\operatorname{rank} H_1\left(J^{C_2-\varepsilon}\right) - \operatorname{rank} H_1\left(J^{C_2-\varepsilon}, J^{-\varepsilon}\right) + 1 \\ &-\operatorname{rank} H_0\left(J^{C_2-\varepsilon}\right) + \operatorname{rank} H_0\left(J^{C_2-\varepsilon}, J^{-\varepsilon}\right) \\ = {}& 0 \end{aligned} \tag{8.96}$$

via (8.67). Inserting (8.94) into (8.96), we have

$$\operatorname{rank} H_0\left(J^{C_2-\varepsilon}\right) = \operatorname{rank} H_1\left(J^{C_2-\varepsilon}\right) + 1 \tag{8.97}$$

and consequently, we deduce from (8.93) that

$$H_1\left(J^{C_2-\varepsilon}\right) \cong G. \tag{8.98}$$

On the other side,

$$\begin{aligned} H_1\left(J^{C_2-\varepsilon}\right) &\cong H_2\left(E, J^{C_2-\varepsilon}\right) \\ &\cong H_2\left(J^{C_2+\varepsilon}, J^{C_2-\varepsilon}\right) \\ &\cong C_2\left(J, u_2\right) \oplus C_2\left(J, u_3\right). \end{aligned} \tag{8.99}$$

The combination of (8.98) and (8.99) also alludes to $C_q\left(J, u_3\right) \cong \delta_{q2} G$, giving rise to a desired contradiction. That's the precise statement (8.58).

We now remain to prove $C_2\left(J, u_3\right) \neq 0$. Indeed, notice that $\inf\limits_{u \in K_J} J(u) \geq 0$, so $C_q(J, \infty) \cong H_q\left(E, J^{-\varepsilon}\right) \cong \delta_{q d_k} G$ for $\forall \varepsilon>0$. Take $\varepsilon>0$ small suitably such that $J^{\varepsilon} \cap K_J=\{0\}$.

Notice that $C_1=J\left(u_1\right)<C_2=J\left(u_2\right)<C_3=J\left(u_3\right)$. Take $\varepsilon_1, \varepsilon_2>0$, s.t., $C_1+\varepsilon_1<C_2<C_2+\varepsilon_2<C_3$. Thanks to $d_k \geq 3$, the combination of (8.60) and (8.61) shows $H_1\left(J^{\varepsilon}\right) \cong 0$.

Since $C_q\left(J, u_1\right) \cong H_q\left(J^{C_1+\varepsilon_1}, J^{\varepsilon}\right) \cong \delta_{q 1} G$, by the following exactness of singular homology groups

$$\begin{aligned} \cdots \rightarrow H_1\left(J^{\varepsilon}\right) &\stackrel{i_*}{\rightarrow} H_1\left(J^{C_1+\varepsilon_1}\right) \stackrel{j_*}{\rightarrow} H_1\left(J^{C_1+\varepsilon_1}, J^{\varepsilon}\right) \\ &\stackrel{\partial}{\rightarrow} H_0\left(J^{\varepsilon}\right) \stackrel{i_*}{\rightarrow} H_0\left(J^{C_1+\varepsilon_1}\right) \stackrel{j_*}{\rightarrow} H_0\left(J^{C_1+\varepsilon_1}, J^{\varepsilon}\right) \\ &\rightarrow \cdots, \end{aligned} \tag{8.100}$$

we derive

$$\begin{aligned} &\operatorname{rank} H_1\left(J^{C_1+\varepsilon_1}\right)-\operatorname{rank} H_1\left(J^{C_1+\varepsilon_1}, J^{\varepsilon}\right) \\ &+\operatorname{rank} H_0\left(J^{\varepsilon}\right)-\operatorname{rank} H_0\left(J^{C_1+\varepsilon_1}\right) \\ =\,&0 \end{aligned} \tag{8.101}$$

and this gets

$$\operatorname{rank} H_0\left(J^{C_1+\varepsilon_1}\right)=\operatorname{rank} H_1\left(J^{C_1+\varepsilon_1}\right)+1. \tag{8.102}$$

Argue by contradiction, $C_2\left(J, u_3\right) \cong 0$, yielding

$$H_2\left(E, J^{C_2+\varepsilon_2}\right) \cong 0. \tag{8.103}$$

As $m\left(u_3\right) \geq n\left(u_3\right) \geq 2$, it follows that $H_1\left(E, J^{C_2+\varepsilon_2}\right) \cong H_0\left(E, J^{C_2+\varepsilon_2}\right) \cong 0$. Observe that $C_q\left(J, u_2\right) \cong H_q\left(J^{C_2+\varepsilon_2}, J^{C_1+\varepsilon_1}\right) \cong \delta_{q 1} G$. Due to the following exact sequence:

$$\begin{aligned} \cdots \rightarrow H_2\left(J^{C_2+\varepsilon_2}, J^{C_1+\varepsilon_1}\right) &\stackrel{\partial}{\rightarrow} H_1\left(J^{C_1+\varepsilon_1}\right) \\ &\stackrel{i_*}{\rightarrow} H_1\left(J^{C_2+\varepsilon_2}\right) \stackrel{j_*}{\rightarrow} H_1\left(J^{C_2+\varepsilon_2}, J^{C_1+\varepsilon_1}\right) \\ &\stackrel{\partial}{\rightarrow} H_0\left(J^{C_1+\varepsilon_1}\right) \stackrel{i_*}{\rightarrow} H_0\left(J^{C_2+\varepsilon_2}\right) \stackrel{j_*}{\rightarrow} H_0\left(J^{C_2+\varepsilon_2}, J^{C_1+\varepsilon_1}\right) \\ &\rightarrow \cdots, \end{aligned} \tag{8.104}$$

we have

$$\begin{aligned}&\operatorname{rank}H_1\left(J^{C_1+\varepsilon_1}\right)-\operatorname{rank}H_1\left(J^{C_2+\varepsilon_2}\right)+1\\&-\operatorname{rank}H_0\left(J^{C_1+\varepsilon_1}\right)+\operatorname{rank}H_0\left(J^{C_2+\varepsilon_2}\right)\\=\,&0.\end{aligned}\tag{8.105}$$

Inserting (8.102) into (8.105),

$$\operatorname{rank}H_1\left(J^{C_2+\varepsilon_2}\right)=\operatorname{rank}H_0\left(J^{C_2+\varepsilon_2}\right)\geq 1,\tag{8.106}$$

violating

$$H_1\left(J^{C_2+\varepsilon_2}\right)\cong H_2\left(E,J^{C_2+\varepsilon_2}\right)\cong C_2\left(J,u_3\right)\cong 0.\tag{8.107}$$

Hence, $C_2(J,u_3)$ is nontrivial, and by Proposition 7.3 $C_2(J,u_3)\cong\delta_{q2}G$ since $m(u_3)\geq 2$. Employing Morse inequality, there exists a nontrivial solution u^* of (1.1) with $C_{d_k}(J,u^*)\neq 0$. Out of the question!

Quite similarly we can treat the case $C_1=C_2$. We complete the proof of Theorem 1.1 thereby.

9. Appendix

Lemma 9.1. *Under the hypotheses of Lemma 2.11, then $\zeta_k=\xi_k$ for $k\leq d_l+1$, where the definitions of ζ_k and ξ_k are presented by (2.28) and (2.29) respectively.*

Proof. Clearly $\xi_k\leq\zeta_k$, so we concentrate on the converse. As $C_0^\infty\left(\mathbb{R}^N\right)$ is dense in $H^1\left(\mathbb{R}^N\right)$, for $w\in E_{k-1}^\perp\cap H^1\left(\mathbb{R}^N\right)$, $\exists u_n\in C_0^\infty\left(\mathbb{R}^N\right)$, $u_n\to w$ in $H^1\left(\mathbb{R}^N\right)$.

We claim that

$$P_{E_{k-1}^\perp}u_n\to w\text{ in }H^1\left(\mathbb{R}^N\right)\tag{9.1}$$

for $k\leq d_l+1$, where $P_{E_{k-1}^\perp}$ is the orthogonal projection onto $E_{k-1}^\perp$ in $L^2\left(\mathbb{R}^N\right)$.

To see this, we first show that $P_{E_{k-1}^\perp}u_n$ is bounded in $H^1\left(\mathbb{R}^N\right)$. In practice, by (3.1)

$$\begin{aligned}\left\|P_{E_{k-1}}u_n\right\|^2&\leq 2\left\|P_{E_{k-1}}u_n\right\|_m^2\\&\leq 2\left(\lambda_l+m\right)\left\|P_{E_{k-1}}u_n\right\|_{L^2}^2\\&\leq 2\left(\lambda_l+m\right)\|u_n\|_{L^2}^2\leq 2\left(\lambda_l+m\right)\|u_n\|^2,\end{aligned}\tag{9.2}$$

and therefore

$$\left\|P_{E_{k-1}^\perp}u_n\right\|\leq\|u_n\|+\left\|P_{E_{k-1}}u_n\right\|\leq\left(1+\left[2\left(\lambda_l+m\right)\right]^{\frac{1}{2}}\right)\|u_n\|.\tag{9.3}$$

Observe that

$$\begin{aligned}\left\| P_{E_{k-1}} u_n \right\|_{L^2} &= \left\| P_{E_{k-1}} (u_n - w) \right\|_{L^2} \\ &\leq \|u_n - w\|_{L^2} \leq \|u_n - w\| \to 0 \end{aligned} \tag{9.4}$$

derives

$$\left\| P_{E_{k-1}} u_n \right\| \to 0, \tag{9.5}$$

consequently,

$$\begin{aligned} &\left\| P_{E_{k-1}^{\perp}} (u_n - w) \right\| - \left\| P_{E_{k-1}} u_n \right\| \\ \leq &\left\| P_{E_{k-1}} u_n + P_{E_{k-1}^{\perp}} (u_n - w) \right\| = \|u_n - w\| \end{aligned} \tag{9.6}$$

yields

$$\left\| P_{E_{k-1}^{\perp}} u_n - w \right\| \leq \left\| P_{E_{k-1}} u_n \right\| + \|u_n - w\| \to 0 \tag{9.7}$$

via (9.5). The claim is thus proved. Thanks to $P_{E_{k-1}} u_n \in H^2\left(\mathbb{R}^N\right)$, $P_{E_{k-1}^{\perp}} u_n \in H^2\left(\mathbb{R}^N\right)$.

The above argument indicates that for $\forall w \in E_{k-1}^{\perp} \cap H^1\left(\mathbb{R}^N\right)$, $\exists w_n \in E_{k-1}^{\perp} \cap H^2\left(\mathbb{R}^N\right)$, $w_n \to w$ in $H^1\left(\mathbb{R}^N\right)$. Hence, there exists a minimizing sequence $\{\varphi_n\}_{n=1}^{+\infty}$ of (2.29), $\varphi_n \in E_{k-1}^{\perp} \cap H^2$, $\|\varphi_n\|_{L^2} = 1$, s.t., $\varepsilon_n > 0$, $\varepsilon_n \to 0$, and

$$\zeta_k \leq \int_{\mathbb{R}^N} |\nabla \varphi_n|^2 + V(x) \varphi_n^2 \leq \xi_k + \varepsilon_n, \tag{9.8}$$

deriving $\zeta_k \leq \xi_k$. We arrive at the conclusion. □

A more general case can be stated as follows:

Corollary 9.2. *Let V be a real K–R potential, then for $\forall \mathfrak{B}_{k-1} \subset H^2$, $\dim \mathfrak{B}_{k-1} = k - 1$,*

$$\inf_{\substack{\psi \in \mathfrak{B}_{k-1}^{\perp} \cap H^1 \\ \|\psi\|_{L^2} = 1}} \int_{\mathbb{R}^N} |\nabla \psi|^2 + V(x) \psi^2 = \inf_{\substack{\psi \in \mathfrak{B}_{k-1}^{\perp} \cap H^2 \\ \|\psi\|_{L^2} = 1}} \langle A\psi, \psi \rangle_{L^2}. \tag{9.9}$$

Proof. Note that there exist $C_1, C_2 > 0$, s.t., for $\forall u \in \mathfrak{B}_{k-1}$,

$$C_1 \|u\|_{L^2} \leq \|u\|_m \leq C_2 \|u\|_{L^2}, \tag{9.10}$$

an argument analogous to Lemma 9.1 ends the proof. □

Lemma 9.3. *Let V be a K–R potential and suppose $\{u_k\}_{k=1}^{+\infty} \subset H^1$, then $u_k \rightharpoonup u_0$ in $H_m^1 \Longleftrightarrow u_k \rightharpoonup u_0$ in H^1.*

Proof. We just deal with the case "⇒" in consideration of similarity argument on the converse. By way of negation, $\exists \varphi^* \in H^1$, $\exists \gamma > 0$,

$$\left|\left\langle u_k - u_0, \varphi^* \right\rangle\right| \geq \gamma. \tag{9.11}$$

As $\{u_k\}_{k=1}^{+\infty}$ is bounded in H^1, there exists a subsequence $\left\{u_{k_j}\right\}_{j=1}^{+\infty}$, s.t. $u_{k_j} \rightharpoonup u^*$ in H^1 as $j \to +\infty$, i.e.

$$\left\langle u_{k_j} - u^*, \varphi \right\rangle \to 0, \forall \varphi \in H^1. \tag{9.12}$$

We claim that $u^* = u_0$ a.e. on $\mathbb{R}^N$. Notice that for fixed $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$, there is a bounded domain $\Omega \subset \mathbb{R}^N$, $\left\{x \in \mathbb{R}^N : \varphi(x) \neq 0\right\} \subset \Omega$. Hence,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N} V(x)\left(u_{k_j} - u^*\right)\varphi \right| \\
&\leq \|V_1\|_{L^p} \cdot \left\|u_{k_j} - u^*\right\|_{L^{2q}(\Omega)} \cdot \|\varphi\|_{L^{2q}(\Omega)} \\
&\quad + \|V_2\|_{L^\infty} \cdot \left\|u_{k_j} - u^*\right\|_{L^2(\Omega)} \cdot \|\varphi\|_{L^2(\Omega)} \\
&\to 0,
\end{aligned} \tag{9.13}$$

$\frac{1}{p} + \frac{1}{q} = 1$, $p > \frac{N}{2}$ if $N \geq 4$, and $p = 2$ if $N \leq 3$.

(9.12) together with (9.13) yields

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla\left(u_{k_j} - u^*\right)\nabla\varphi + (V + m)\left(u_{k_j} - u^*\right)\varphi \\
&= \int_{\mathbb{R}^N} \nabla\left(u_{k_j} - u^*\right)\nabla\varphi + \left(u_{k_j} - u^*\right)\varphi \\
&\quad + \int_{\mathbb{R}^N} V(x)\left(u_{k_j} - u^*\right)\varphi + (m-1)\int_{\mathbb{R}^N}\left(u_{k_j} - u^*\right)\varphi \\
&\to 0,
\end{aligned} \tag{9.14}$$

indicating

$$\left\langle u_{k_j} - u^*, \varphi \right\rangle_m \to 0 \tag{9.15}$$

for $\forall \varphi \in C_0^\infty\left(\mathbb{R}^N\right)$. On the other side, in view of the hypothesis, we have

$$\left\langle u_{k_j} - u_0, \varphi \right\rangle_m \to 0 \tag{9.16}$$

for $\forall \varphi \in C_0^\infty\left(\mathbb{R}^N\right)$.

Using (9.15) and (9.16) shows

$$\langle u^* - u_0, \varphi \rangle_m = 0, \forall \varphi \in C_0^\infty \left(\mathbb{R}^N \right). \tag{9.17}$$

As $C_0^\infty \left(\mathbb{R}^N \right)$ is dense in $H^1 \left(\mathbb{R}^N \right)$, (9.17) derives

$$\langle u^* - u_0, \varphi \rangle_m = 0, \forall \varphi \in H^1 \left(\mathbb{R}^N \right). \tag{9.18}$$

Take $\varphi = u^* - u_0$ and the claim is thus proved. This derives a desired contradiction by combining (9.11) and (9.12). The proof is complete. □

The following proposition is a C^2 version of Theorem 5.6 of [8] and also a variant version of Theorem 8.8 of [29]. The proof is almost the same as that of Theorem 5.6 of [8].

Proposition 9.4. *Let E be a Hilbert space. Suppose that $\left\{ f_\sigma \in C^{2-0}(E, \mathbb{R}) \mid \sigma \in [0,1] \right\}$ is a family of functions satisfying the (PS) condition. Suppose that there exists an open set N such that f_σ has a unique critical point p_σ in N, $\forall \sigma \in [0,1]$, and that $\sigma \to f_\sigma$ is continuous in $C^1\left(\overline{N}\right)$ topology. Then $C_*(f_\sigma, p_\sigma)$ is independent of σ.*

Acknowledgments

We are greatly indebted to Professor Zhaoli Liu for many valuable suggestions to the proofs of Lemmas 7.6 and 7.7. Moreover, the first author would like to sincerely thank his family's work behind the scenes over the years, and also appreciate for strong supports by NSFC (11471319) and BCMIIS (Beijing Center for Mathematics and Information Interdisciplinary Sciences).

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