Convergence of HX Preconditioner for Maxwell's Equations with Jump Coefficients (i): Various Extensions of The Regular Helmholtz Decomposition

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Abstract

This paper is the first one of two serial articles, whose goal is to prove convergence of HX Preconditioner (proposed by Hiptmair and Xu [14]) for Maxwell's equations with jump coefficients. In this paper we establish various extensions of the regular Helmholtz decomposition for edge finite element functions defined in three dimensional domains. The functions defined by the regular Helmholtz decompositions can preserve the zero tangential complement on faces and edges of polyhedral domains and some non-Lipchitz domains, and possess stability estimates with only a *logarithm* factor. These regular Helmholtz decompositions will be used to prove convergence of the HX preconditioner for Maxwell's equations with jump coefficients in another paper [15].

Key Words. Maxwell's equations, Nedelec elements, regular Helmholtz decomposition, stability

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Contents

1 Introduction

The (orthogonal or regular) Helmholtz decomposition says that any vector-valued function in $H(\mathbf{curl})$ space can be decomposed into the sum of a H^1 vector-valued function and the gradient of a H^1 scalar-valued function (refer to [10] and [9]), and the decomposition is stable with respect to the standard norms. The regular Helmholtz decomposition is nicer than the

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orthogonal Helmholtz decomposition in the sense that the regular Helmholtz decomposition is valid on the general Lipchitz domains, but the orthogonal Helmholtz decomposition hold only on smooth domains or convex domains. Own to the Helmholtz decomposition, the problem for a $H(\mathbf{curl})$ functions can be transformed into the problem on two H^1 functions.

The Nedelec edge finite element method (see [24]) is a popular discretization method of Maxwell's equations. The Helmholtz decompositions for the edge finite element functions, which is called discrete Helmholtz decompositions, play a key role in numerical analysis, especially in the convergence analysis of preconditioners, for Maxwell's equations (see, for example, [12, 14, 17, 18, 19, 25, 28, 29]). However, in applications nonhomogeneous medium is often encountered, and so some weighted norms have to be introduced. A natural question is: whether the discrete Helmholtz decomposition is still stable with respect to the weighted norms? It seems not easy to give a positive answer to this question. The first work on Helmholtz decomposition in three dimensional nonhomogeneous medium was done in [17], where a discrete weighted *orthogonal* Helmholtz decomposition was constructed and proved to be almost stable with respect to a weight function.

This paper is the first one of two serial articles. The purposes of the serial articles are to build a discrete weighted *regular* Helmholtz decomposition in three dimensions and to prove the convergence of HX preconditioner [14] for Maxwell's equations with jump coefficients based on the new Helmholtz decomposition. For these purposes, in this paper we first develop some technical tools to derive various extensions of the discrete regular Helmholtz decomposition in three dimensions. The standard regular (and orthogonal) Helmholtz decomposition possesses a very important property: when the considered vector-valued function has zero trace on the boundary of the underlying domain, the functions defined by the Helmholtz decomposition also have the zero trace on this boundary. We will construct discrete *regular* Helmholtz decompositions on polyhedral domains such that the property mentioned above can be kept when the boundary is replaced by a union of some local faces and edges of the polyhedral domain. We can require that the functions defined by the decomposition vanish at any vertex of the polyhedron, provided that the considered vector-valued function satisfies a constraint for each vertex. In particular, we also establish the corresponding Helmholtz decompositions for some non-Lipchitz domains, which are unions of two polyhedral domains whose intersection is just one edge or one vertex. We will show that the regular Helmholtz decompositions possess stability estimates with only a *logarithm* factor. These results, which are of interest themselves, will be used in [15] to develop a discrete weighted regular Helmholtz decomposition, by which the convergence of HX preconditioner for the case with jump coefficients will be further proved.

The outline of the paper is as follows. In Section 2, we define some edge finite element subspaces. In section 3, we prove regular Helmholtz decompositions preserving zero tangential trace on faces. In Section 4, we present several discrete regular Helmholtz decompositions preserving local zero tangential complements on edges and faces. In Section 5, we derive discrete regular Helmholtz decompositions on some non-Lipchitz domains.

2 Preliminaries

This section introduce some fundamental finite element spaces.

2.1 Sobolev spaces and norms

For an open and connected bounded domain G in $\mathbf{R}^3$, let $H^1(G)$ be the standard Sobolev space. Define the **curl**-space as follows

$$H(\mathbf{curl}; G) = \{\mathbf{v} \in L^2(G)^3;\ \mathbf{curl}\,\mathbf{v} \in L^2(G)^3\}.$$

Set

$$\|\mathbf{v}\|_{1,G} = (|\mathbf{v}|_{1,G}^2 + \|\mathbf{v}\|_{0,G}^2)^{\frac{1}{2}} \quad \mathbf{v} \in (H^1(G))^3$$

and

$$\|\mathbf{v}\|_{\mathbf{curl},G} = (\|\mathbf{curl}\ \mathbf{v}\|_{0,G}^2 + \|\mathbf{v}\|_{0,G}^2)^{\frac{1}{2}} \quad \mathbf{v} \in H(\mathbf{curl};\ G).$$

For a (may be non-convex) polyhedron G, let Γ be a (closed) face or the union of several faces of G. Define

$$H_\Gamma(\mathbf{curl};\ G) = \{\mathbf{v} \in H(\mathbf{curl};\ G) :\ \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma\}$$

and

$$H_\Gamma^1(G) = \{v \in H^1(G) :\ v = 0 \text{ on } \Gamma\}.$$

2.2 Edge and nodal element spaces

For a polyhedron G, let G be divided into smaller tetrahedral elements of size h, and let $\mathcal{T}_h$ denote the resulting triangulation of the domain G. As usual, we assume that the triangulation $\mathcal{T}_h$ is quasi-uniform. We use $\mathcal{E}_h$ and $\mathcal{N}_h$ to denote the set of edges of $\mathcal{T}_h$ and the set of nodes in $\mathcal{T}_h$ respectively. Then the Nedelec edge element space, of the lowest order, is a subspace of piecewise linear polynomials defined on $\mathcal{T}_h$:

$$V_h(G) = \Big\{\mathbf{v} \in H(\mathbf{curl};\ G);\ \mathbf{v}|_K \in R(K),\ \forall K \in \mathcal{T}_h\Big\},$$

where $R(K)$ is a subset of all linear polynomials on the element K of the form:

$$R(K) = \Big\{\mathbf{a} + \mathbf{b} \times \mathbf{x};\ \mathbf{a}, \mathbf{b} \in \mathbf{R}^3,\ \mathbf{x} \in K\Big\}.$$

It is known that, for any $\mathbf{v} \in V_h(G)$, its tangential components are continuous on all edges in $\mathcal{E}_h$, and $\mathbf{v}$ is uniquely determined by its moments on each edge e of $\mathcal{T}_h$:

$$M_h(\mathbf{v}) = \Big\{\lambda_e(\mathbf{v}) = \int_e \mathbf{v} \cdot \mathbf{t}_e ds;\ e \in \mathcal{E}_h\Big\}$$

where $\mathbf{t}_e$ denotes the unit vector on edge e, and this notation will be used to denote any edge or union of edges, either from an element $K \in \mathcal{T}_h$ or from G itself. For example, for a face F of G, the notation $\mathbf{t}_{\partial \mathrm{F}}$ denotes the unit vector along $\partial \mathrm{F}$. For a vector-valued function $\mathbf{v}$ with appropriate smoothness, we introduce its edge element interpolation $\mathbf{r}_h \mathbf{v}$ such that $\mathbf{r}_h \mathbf{v} \in V_h(G)$, and $\mathbf{r}_h \mathbf{v}$ and $\mathbf{v}$ have the same moments as in $M_h(\mathbf{v})$. The interpolation operator $\mathbf{r}_h$ will be used in the construction of a stable decomposition for any function $\mathbf{v}_h \in V_h(G)$.

As we will see, the edge element analysis involves also frequently the nodal element space. For this purpose we introduce $Z_h(G)$ to be the standard continuous piecewise linear finite element space in $H^1(G)$ associated with the triangulation $\mathcal{T}_h$.

Define

$$V_h(\partial G) = \{(\mathbf{v} \times \mathbf{n})|_{\partial G};\ \mathbf{v} \in V_h(G)\},$$

$$Z_h^0(G) = \{q \in Z_h(G);\ q = 0 \text{ on } \partial G\}$$

and

$$V_h^0(G) = \{\mathbf{v} \in V_h(G);\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial G\}.$$

Throughout this subsection, we shall consider a (may be non-convex) polyhedron G. We will often use F, E and V to denote a general face, edge and vertex of G respectively, but use e to denote a general edge of $\mathcal{T}_h$ lying on ∂G.

From now on, we shall frequently use the notations $\lesssim$ and $\eqsim$. For any two non-negative quantities x and y, $x \lesssim y$ means that $x \leq Cy$ for some constant C independent of mesh size h, subdomain size d and the possible large jumps of some related coefficient functions across the interface between any two subdomains. $x \eqsim y$ means $x \lesssim y$ and $y \lesssim x$.

3 Regular Helmholtz decompositions preserving zero tangential trace on faces

In this section we develop regular Helmholtz decompositions for vector-valued functions that have zero tangential trace on some faces of a polyhedron. We use the notations introduced in the previous section.

Lemma 3.1 *For any $\mathbf{v} \in H_\Gamma(\mathbf{curl};\ G)$, there exists a vector-valued function $\Phi \in (H_\Gamma^1(G))^3$ and a scalar function $p \in H_\Gamma^1(G)$ such that*

$$\mathbf{v} = \Phi + \nabla p. \tag{3.1}$$

Moreover, we have

$$\|\Phi\|_{1,G} \lesssim \|\mathbf{curl}\,\mathbf{v}\|_{0,G} \quad \text{and} \quad \|p\|_{1,G} \lesssim \|\mathbf{v}\|_{\mathbf{curl},G}. \tag{3.2}$$

In particular, if $\mathbf{v} = \mathbf{v}_h \in V_h(G)$, then

$$\mathbf{v}_h = \mathbf{r}_h\Phi + \nabla p_h \tag{3.3}$$

with $p_h \in Z_h(G) \cap H_\Gamma^1(G)$, which satisfies

$$\|p_h\|_{1,G} \lesssim \|\mathbf{v}\|_{\mathbf{curl},G}. \tag{3.4}$$

Moreover, there exist $\Phi_h \in (Z_h(G) \cap H_\Gamma^1(G))^3$ and $\mathbf{R}_h \in V_h(G) \cap H_\Gamma(\mathbf{curl};\ G)$ such that

$$\mathbf{v}_h = \mathbf{r}_h\Phi_h + \nabla p_h + \mathbf{R}_h, \tag{3.5}$$

and

$$\|\Phi_h\|_{1,G} \lesssim \|\mathbf{curl}\,\mathbf{v}\|_{0,G} \quad \text{and} \quad h^{-1}\|\mathbf{R}_h\|_{0,G} \lesssim \|\mathbf{curl}\,\mathbf{v}\|_{0,G}. \tag{3.6}$$

When either G is convex or Γ contains the concave part of ∂G, the functions Φ, p, Φ_h and p_h defined above satisfy the estimates

$$\|\Phi\|_{1,G} \lesssim \|\mathbf{curl}\,\mathbf{v}\|_{0,G} \quad \text{and} \quad \|\Phi\|_{0,G} + \|p\|_{1,G} \lesssim \|\mathbf{v}\|_{0,G} \tag{3.7}$$

and

$$\|\Phi_h\|_{1,G} \lesssim \|\mathbf{curl}\,\mathbf{v}\|_{0,G} \quad \text{and} \quad \|\Phi_h\|_{0,G} + \|p_h\|_{1,G} \lesssim \|\mathbf{v}\|_{0,G}. \tag{3.8}$$

Proof. The proof follows the arguments in [9, 25]. Let B be a polyhedron domain containing G as its subdomain such that $\partial G \cap \partial B = \partial G \backslash \Gamma$ and the size of the complement $D = B \backslash G$ is a positive number independent of h. It is easy to see that $\partial G \cap \partial D = \Gamma$. We extend $\mathbf{v}$ onto the global B by zero, i.e., the extension $\tilde{\mathbf{v}}$ satisfying $\tilde{\mathbf{v}} = \mathbf{v}$ on G and $\tilde{\mathbf{v}} = \mathbf{0}$ on $\bar{D}$. Since $\mathbf{v} \times \mathbf{n} = \mathbf{0}$ on Γ, we have $\tilde{\mathbf{v}} \in H(\mathbf{curl};\ B)$. By the regular Helmholtz decomposition (Lemma 2.4 in [13]), we get

$$\tilde{\mathbf{v}} = \mathbf{w} + \nabla \varphi \quad \text{on } B, \tag{3.9}$$

with $\mathbf{w} \in (H^1(B))^3$ and $\varphi \in H^1(B)/\mathbb{R}$. Moreover, $\mathbf{w}$ and φ satisfy

$$\|\mathbf{w}\|_{1,B} \lesssim \|\mathbf{curl}\,\tilde{\mathbf{v}}\|_{0,B} = \|\mathbf{curl}\,\mathbf{v}\|_{0,G} \ \text{ and } \ \|\varphi\|_{1,B} \lesssim \|\tilde{\mathbf{v}}\|_{\mathbf{curl},B} = \|\mathbf{v}\|_{\mathbf{curl},G}. \tag{3.10}$$

When either G is convex or Γ contains the concave part of ∂G, we can require that B is also convex. Then, by the orthogonal Helmholtz decomposition in [10], we have

$$\|\mathbf{w}\|_{1,B} \lesssim \|\mathbf{curl}\,\mathbf{v}\|_{0,G} \ \text{ and } \ \|\mathbf{w}\|_{0,B} + \|\varphi\|_{1,B} \lesssim \|\mathbf{v}\|_{0,G}. \tag{3.11}$$

Noting that $\tilde{\mathbf{v}} = \mathbf{0}$ on $\bar{D}$, we have $\nabla \varphi = -\mathbf{w}$ on D, and so $\varphi \in H^2(D)$. Let $\tilde{\varphi} \in H^2(B)$ be the stable extension of φ from D onto the global B. It follows by (3.9) that

$$\tilde{\mathbf{v}} = (\mathbf{w} + \nabla \tilde{\varphi}) + \nabla(\varphi - \tilde{\varphi}) \quad \text{on } B. \tag{3.12}$$

Define $\Phi = \mathbf{w} + \nabla \tilde{\varphi}$ and $p = \varphi - \tilde{\varphi}$. Then we have $\Phi \in (H^1(B)^3$ and $p \in H^1(B)$, and they satisfy (3.1). Since $\Phi = \mathbf{0}$ on $\bar{D}$, we obtain $\Phi = \mathbf{0}$ on Γ, which implies that $(\nabla p) \times \mathbf{n} = (\mathbf{v} - \Phi) \times \mathbf{n} = \mathbf{0}$ on Γ. Therefore $\Phi \in (H^1_\Gamma(G))^3$ and $p \in H^1_\Gamma(G)$.

By the definition of Φ and the first inequality in (3.10), we get

$$\begin{aligned} \|\Phi\|_{1,G} &\leq \|\mathbf{w}\|_{1,G} + \|\tilde{\varphi}\|_{2,G} \\ &\lesssim \|\mathbf{curl}\,\mathbf{v}\|_{0,G} + \|\varphi\|_{2,D} \\ &\lesssim \|\mathbf{curl}\,\mathbf{v}\|_{0,G} + \|\mathbf{w}\|_{1,D} \\ &\lesssim \|\mathbf{curl}\,\mathbf{v}\|_{0,G} + \|\mathbf{w}\|_{1,B} \\ &\lesssim \|\mathbf{curl}\,\mathbf{v}\|_{0,G}. \end{aligned}$$

We can further obtain the second inequality of (3.2) by (3.1). The decompositions (3.3) can be obtained by the property of the interpolation operator $\mathbf{r}_h$.

Let $\mathbf{Q}_h : (H^1_\Gamma(G))^3 \rightarrow (Z_h(G) \cap H^1_\Gamma(G))^3$ denote the L^2 projector. Define $\Phi_h = \mathbf{Q}_h \Phi$ and

$$\mathbf{R}_h = \mathbf{r}_h(\mathbf{I} - \mathbf{Q}_h)\Phi = (\mathbf{I} - \mathbf{Q}_h)\Phi + (\mathbf{r}_h - \mathbf{I})(\mathbf{I} - \mathbf{Q}_h)\Phi.$$

Notice that both $\mathbf{Q}_h$ and $\mathbf{r}_h$ possess the optimal L^2 approximation on the space $(H^1(G))^3$, the estimate (3.6) can be derived immediately. The estimates (3.7) and (3.8) can be verified similarly by using (3.11). ♯

Remark 3.1 *The key tool in the above proof is the well known extension theorem for the considered domain. The extension theorem was extended to more general domain in [20]. This kind of domain, which is called (ε, δ) domain or Jone domain, can be highly non-rectifiable and no regularity condition on its boundary, and includes the classical snowflake domain of conformal mapping theory and small perturbation domain of a polyhedron, where some face of the perturbation domain is a union of faces of some elements and is not a plane face. Thus Theorem 3.1 is also valid for Jone domain, in which Γ may be heavily irregular.*

As we will see in [15], the second inequality in (3.8) will play key role in the analysis for the case that the zero order term in the considered Maxwell equations is dominated. Unfortunately, the results do not hold yet when G is a non-convex polyhedron (unless Γ contains the concave part of ∂G), cf. Remark 4 in [14]. In the following we establish slightly weaker results for such situation. To this end, we first give a simple auxiliary result.

Proposition 3.1. Let G be a polyhedron, and assume that $\mathbf{w}_h \in (Z_h(G))^3$. Then we have $\mathbf{curl}\ (\mathbf{r}_h\mathbf{w}_h) = \mathbf{curl}\ \mathbf{w}_h$ and $\|\mathbf{r}_h\mathbf{w}_h\|_{0,G} \lesssim \|\mathbf{w}_h\|_{0,G}$.

Proof. Let $W_h(G)$ denote the Raviart-Thomas finite element space of the lowest order, and let Π_h be the interpolation operator into $W_h(G)$. Since $\mathbf{w}_h \in (Z_h(G))^3$, we have $\mathbf{curl}\ \mathbf{w}_h \in W_h(G)$. Then

$$\mathbf{curl}\ (\mathbf{r}_h\mathbf{w}_h) = \Pi_h \mathbf{curl}\ \mathbf{w}_h = \mathbf{curl}\ \mathbf{w}_h.$$

The desired inequality can be derived by the approximation property of $\mathbf{r}_h$ and the inverse estimate of finite element functions. ♯

The following results give a slightly weak L^2 stability of the regular Helmholtz decomposition for the case of non-convex polyhedron.

Theorem 3.1 *Let G be a non-convex polyhedron, which is a union of several convex polyhedra, and let Γ be a union of several faces of G. Assume that $\mathbf{v}_h \in V_h(G)$ satisfies $\mathbf{v}_h \times \mathbf{n} = \mathbf{0}$ on Γ. Then there exist $p_h \in Z_h(G) \cap H^1_\Gamma(G)$, $\mathbf{w}_h \in (Z_h(G) \cap H^1_\Gamma(G))^3$ and $\mathbf{R}_h \in V_h(G) \cap H_\Gamma(\mathbf{curl};\ G)$ such that*

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h\mathbf{w}_h + \mathbf{R}_h, \tag{3.13}$$

with the following estimates

$$\|\mathbf{w}_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}, \quad \|\mathbf{w}_h\|_{0,G} + \|p_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{v}_h\|_{0,G} \tag{3.14}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{0,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}. \tag{3.15}$$

Proof. Without loss of generality, we assume that G is the union of three cubes: $G = D_1 \cup D_2 \cup D_3$ with $D_1 = [0, \frac{1}{2}]^3$, $D_2 = [\frac{1}{2}, 1] \times [0, \frac{1}{2}]^2$ and $D_3 = [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \times [0, \frac{1}{2}]$.

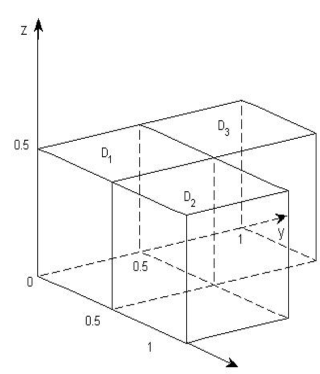

Figure 1: A non-convex polyhedron composed of three cubes

We divide the proof into two steps.

Step 1. Build the desired decomposition.

We first build a decomposition of $\mathbf{v}_h$ on D_1 by Lemma 3.1. Since D_1 is convex, the function $\mathbf{v}_{h,1}=\mathbf{v}_h|_{D_1}$ admits the decomposition

$$\mathbf{v}_{h,1}=\nabla p_{h,1}+\mathbf{r}_h\mathbf{w}_{h,1}+\mathbf{R}_{h,1} \quad \text{on } D_1, \tag{3.16}$$

with $p_{h,1}\in Z_h(D_1)$, $\mathbf{w}_{h,1}\in(Z_h(D_1))^3$ and $\mathbf{R}_{h,1}\in V_h(D_1)$, which satisfy $p_{h,1}=0$, $\mathbf{w}_{h,1}=\mathbf{0}$ and $\mathbf{R}_{h,1}\times\mathbf{n}=0$ on $\partial D_1\cap\Gamma$ (when $\partial D_1\cap\Gamma=\emptyset$, we can require that $p_{h,1}$ has the zero average value on D_1). Moreover, we have

$$\|\mathbf{w}_{h,1}\|_{1,D_1}\lesssim\|\mathbf{curl}\,\mathbf{v}_{h,1}\|_{0,D_1},\quad \|\mathbf{w}_{h,1}\|_{0,D_1}+\|p_{h,1}\|_{1,D_1}\lesssim\|\mathbf{v}_{h,1}\|_{0,D_1} \tag{3.17}$$

and

$$h^{-1}\|\mathbf{R}_{h,1}\|_{0,D_1}\lesssim\|\mathbf{curl}\,\mathbf{v}_{h,1}\|_{0,D_1}. \tag{3.18}$$

Secondly, we extend $\mathbf{w}_{h,1}$ and $p_{h,1}$ into D_2 and D_3 in a special manner such that some stability can be satisfied.

For $k=2,3$, set $\mathrm{F}_{1k}=\partial D_1\cap\partial D_k$, and let $\vartheta_{\mathrm{F}_{1k}}$ be the finite element function defined in [6] and [30]. This function satisfies $\vartheta_{\mathrm{F}_{1k}}(\mathbf{x})=1$ for each node $\mathbf{x}\in\bar{\mathrm{F}}_{1k}\backslash\partial\mathrm{F}_{1k}$, $\vartheta_{\mathrm{F}_{1k}}(\mathbf{x})=0$ for $\mathbf{x}\in\partial D_1\backslash\mathrm{F}_{1k}$ and $0\leq\vartheta_{\mathrm{F}_{1k}}\leq 1$ on D_1. Let π_h denote the standard interpolation operator into $Z_h(D_1)$, and define $\mathbf{w}_{h,1}^{\mathrm{F}_{1k}}=\pi_h(\vartheta_{\mathrm{F}_{1k}}\mathbf{w}_{h,1})$ $(k=2,3)$. Then $\mathbf{w}_{h,1}^{\mathrm{F}_{1k}}\in(Z_h(D_1))^3$ and $\mathbf{w}_{h,1}^{\mathrm{F}_{1k}}=\mathbf{0}$ on $\partial D_1\backslash\mathrm{F}_{1k}$ $(k=2,3)$. By the extension theorem and the Scott-Zhang interpolation [27] , we can show (refer to the proof of Lemma 4.5 in [21]) there exists an extension $\tilde{\mathbf{w}}_{h,1}^{\mathrm{F}_{1k}}\in(Z_h(G))^3$ such that $\tilde{\mathbf{w}}_{h,1}^{\mathrm{F}_{1k}}=\mathbf{w}_{h,1}^{\mathrm{F}_{1k}}$ on D_1, $\tilde{\mathbf{w}}_{h,1}^{\mathrm{F}_{1k}}$ vanishes on $\partial D_k\backslash\mathrm{F}_{1k}$ $(k=2,3)$ and satisfies

$$\|\tilde{\mathbf{w}}_{h,1}^{\mathrm{F}_{1k}}\|_{1,D_k}\lesssim\|\mathbf{w}_{h,1}^{\mathrm{F}_{1k}}\|_{1,D_1}\quad\text{and}\quad\|\tilde{\mathbf{w}}_{h,1}^{\mathrm{F}_{1k}}\|_{0,D_k}\lesssim\|\mathbf{w}_{h,1}^{\mathrm{F}_{1k}}\|_{0,D_1}\quad(k=2,3). \tag{3.19}$$

Set $\mathrm{F}^{\partial}=\partial\mathrm{F}_{12}\cup\partial\mathrm{F}_{13}$, and let $\tilde{\mathbf{w}}_{h,1}^{\partial}\in(Z_h(G))^3$ denote the natural zero extension of $\mathbf{w}_{h,1}|_{\mathrm{F}^{\partial}}$. Define $\tilde{\mathbf{w}}_{h,1}$ as follows:

$$\tilde{\mathbf{w}}_{h,1}=\mathbf{w}_{h,1}\ \text{ on }\ D_1;\quad \tilde{\mathbf{w}}_{h,1}=\tilde{\mathbf{w}}_{h,1}^{\mathrm{F}_{1k}}+\tilde{\mathbf{w}}_{h,1}^{\partial}|_{D_k}\quad\text{on}\quad D_k\ \ (k=2,3).$$

It is easy to see that $\tilde{\mathbf{w}}_{h,1}\in(Z_h(G))^3$.

We define the extension $\tilde{p}_{h,1}\in Z_h(G)$ as follows: $\tilde{p}_{h,1}=p_{h,1}$ on $\bar{D}_1$; $\tilde{p}_{h,1}$ vanishes at all the nodes in $\partial D_k\backslash\bar{\mathrm{F}}_{1k}$ $(k=2,3)$; $\tilde{p}_{h,1}$ is discrete harmonic in D_k $(k=2,3)$. Let $\tilde{\mathbf{R}}_{h,1}\in V_h(G)$ be the natural zero extension of $\mathbf{R}_{h,1}$. For $k=2,3$, we define

$$\mathbf{v}_{h,k}^{*}=\mathbf{v}_h|_{D_k}-(\nabla\tilde{p}_{h,1}+\mathbf{r}_h\tilde{\mathbf{w}}_{h,1}+\tilde{\mathbf{R}}_{h,1})|_{D_k}\quad\text{on}\quad D_k. \tag{3.20}$$

It is easy to see that $\mathbf{v}_{h,k}^{*}\times\mathbf{n}=\mathbf{0}$ on $\bar{\mathrm{F}}_{1k}\cup(\partial D_k\cap\Gamma)$ $(k=2,3)$.

Now we build the desired decomposition based on a Helmholtz decomposition of the function $\mathbf{v}_{h,k}^{*}$ $(k=2,3)$ defined above.

Notice that D_k is a convex polyhedron. It follows by Lemma 3.1 that the function $\mathbf{v}_{h,k}^{*}$ admits the decomposition

$$\mathbf{v}_{h,k}^{*}=\nabla p_{h,k}^{*}+\mathbf{r}_h\mathbf{w}_{h,k}^{*}+\mathbf{R}_{h,k}^{*}\quad\text{on } D_k\quad(k=2,3), \tag{3.21}$$

with $p_{h,k}^{*}\in Z_h(D_k)$, $\mathbf{w}_{h,k}^{*}\in(Z_h(D_k))^3$ and $\mathbf{R}_{h,k}^{*}\in V_h(D_k)$ $(k=2,3)$, which satisfy $p_{h,k}^{*}=0$, $\mathbf{w}_{h,k}^{*}=\mathbf{0}$ and $\mathbf{R}_{h,k}^{*}\times\mathbf{n}=0$ on $\bar{\mathrm{F}}_{1k}\cup(\partial D_k\cap\Gamma)$ $(k=2,3)$. Moreover, for $k=2,3$ we have

$$\|\mathbf{w}_{h,k}^{*}\|_{1,D_k}\lesssim\|\mathbf{curl}\,\mathbf{v}_{h,k}^{*}\|_{0,D_k},\quad\|\mathbf{w}_{h,k}^{*}\|_{0,D_k}+\|p_{h,k}^{*}\|_{1,D_k}\lesssim\|\mathbf{v}_{h,k}^{*}\|_{0,D_k} \tag{3.22}$$

and

$$h^{-1}\|\mathbf{R}^*_{h,k}\|_{0,D_k} \lesssim \|\mathbf{curl}\,\mathbf{v}^*_{h,k}\|_{0,D_k}. \tag{3.23}$$

Since $p^*_{h,k}$, $\mathbf{w}^*_{h,k}$ and $\mathbf{R}^*_{h,k}$ have the zero degrees of freedom on $\bar{\mathrm{F}}_{1k}$, we can naturally extend them into G by zero. We denote the resulting zero extentions by $\tilde{p}^*_{h,k}$, $\tilde{\mathbf{w}}^*_{h,k}$ and $\tilde{\mathbf{R}}^*_{h,k}$. Define

$$p_h = \tilde{p}_{h,1} + \sum_{k=2}^{3} \tilde{p}^*_{h,k}, \quad \mathbf{w}_h = \tilde{\mathbf{w}}_{h,1} + \sum_{k=2}^{3} \tilde{\mathbf{w}}^*_{h,k} \text{ and } \mathbf{R}_h = \tilde{\mathbf{R}}_{h,1} + \sum_{k=2}^{3} \tilde{\mathbf{R}}^*_{h,k}.$$

It is easy to see that p_h, $\mathbf{w}_h$ and $\mathbf{R}_h$ have the zero degrees of freedom on Γ. Using the local decompositions (3.16) and (3.21), together with the relation (3.20), we get the global decomposition of $\mathbf{v}_h$

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h. \tag{3.24}$$

Step 2. Derive the stability estimates.

From the definition of $\mathbf{w}_h$, we have

$$\|\mathbf{w}_h\|_{1,G} \lesssim \|\tilde{\mathbf{w}}_{h,1}\|_{1,G} + \sum_{k=2}^{3} \|\tilde{\mathbf{w}}^*_{h,k}\|_{1,D_k} \tag{3.25}$$

For $k = 2, 3$, by (3.22) and (3.20) we can deduce that

$$\|\tilde{\mathbf{w}}^*_{h,k}\|_{1,D_k} \lesssim \|\mathbf{curl}\,\mathbf{v}^*_{h,k}\|_{0,D_k} \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,D_k} + \|\mathbf{curl}(\mathbf{r}_h\tilde{\mathbf{w}}_{h,1})\|_{0,D_k} + \|\mathbf{curl}\,\tilde{\mathbf{R}}_{h,1}\|_{0,D_k}.$$

Applying **Proposition 3.1** and the inverse estimate to the last two norms in the above inequality, we get

$$\begin{aligned} \|\tilde{\mathbf{w}}^*_{h,k}\|_{1,D_k} &\lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,D_k} + \|\mathbf{curl}\,\tilde{\mathbf{w}}_{h,1}\|_{0,D_k} + h^{-1}\|\tilde{\mathbf{R}}_{h,1}\|_{0,D_k} \\ &\lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,D_k} + \|\tilde{\mathbf{w}}_{h,1}\|_{1,D_k} + h^{-1}\|\mathbf{R}_{h,1}\|_{0,D_1}. \end{aligned} \tag{3.26}$$

Here we have used the relation $\|\tilde{\mathbf{R}}_{h,1}\|_{0,D_k} \lesssim \|\mathbf{R}_{h,1}\|_{0,D_1}$, which can be verified directly by the definition of $\tilde{\mathbf{R}}_{h,1}$. Substituting (3.26) into (3.25), and using (3.17)-(3.18), yields

$$\|\mathbf{w}_h\|_{1,G} \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,G} + \sum_{k=2}^{3} \|\tilde{\mathbf{w}}_{h,1}\|_{1,D_k}. \tag{3.27}$$

Similarly, we can show

$$\|\mathbf{w}_h\|_{0,G} \lesssim \|\mathbf{v}_h\|_{0,G} + \sum_{k=2}^{3} \|\tilde{\mathbf{w}}_{h,1}\|_{0,D_k}. \tag{3.28}$$

It suffices to estimate $\|\tilde{\mathbf{w}}_{h,1}\|_{1,D_k}$ and $\|\tilde{\mathbf{w}}_{h,1}\|_{0,D_k}$ $(k = 2, 3)$. By using Lemma 3.36 in [6] and Lemma 4.24 in [30], we get for $k = 2, 3$

$$\|\mathbf{w}^{\mathrm{F}_{1k}}_{h,1}\|_{1,D_1} \lesssim \log(1/h)\|\mathbf{w}_{h,1}\|_{1,D_1}.$$

Combining this inequality with (3.19), leads to

$$\|\tilde{\mathbf{w}}^{\mathrm{F}_{1k}}_{h,1}\|_{1,D_k} \lesssim \log(1/h)\|\mathbf{w}_{h,1}\|_{1,D_1}. \tag{3.29}$$

On the other hand, from the "edge" lemma (cf.[30] and [31]), we have

$$\|\tilde{\mathbf{w}}^{\partial}_{h,1}\|_{1,D_k} \lesssim \|\mathbf{w}_{h,1}\|_{0,\mathrm{F}_{1k}} \lesssim \log^{\frac{1}{2}}(1/h)\|\mathbf{w}_{h,1}\|_{1,D_1} \quad (k=2,3).$$

By the definition of $\tilde{\mathbf{w}}_{h,1}$, together with (3.29) and the above estimate, we deduce that

$$\|\tilde{\mathbf{w}}_{h,1}\|_{1,D_k} \lesssim \log(1/h)\|\mathbf{w}_{h,1}\|_{1,D_1} \quad (k=2,3).$$

Plugging this into (3.27), gives the first estimate in (3.14).

It is easy to see, from the definitions of $\mathbf{w}^{\mathrm{F}_{1k}}_{h,1}$ and $\tilde{\mathbf{w}}^{\partial}_{h,1}$, that

$$\|\mathbf{w}^{\mathrm{F}_{1k}}_{h,1}\|_{0,D_1} \lesssim \|\mathbf{w}_{h,1}\|_{0,D_1} \quad \text{and} \quad \|\tilde{\mathbf{w}}^{\partial}_{h,1}\|_{0,D_k} \lesssim \|\mathbf{w}_{h,1}\|_{0,D_1} \quad (k=2,3). \tag{3.30}$$

This, together with the second inequality in (3.19), leads to

$$\|\tilde{\mathbf{w}}_{h,1}\|_{0,D_k} \lesssim \|\mathbf{w}_{h,1}\|_{0,D_1} \quad (k=2,3). \tag{3.31}$$

Substituting this into (3.28), yields

$$\|\mathbf{w}_h\|_{0,G} \lesssim \|\mathbf{v}_h\|_{0,G}. \tag{3.32}$$

In the following we estimate $\|p_h\|_{1,G}$. If suffices to consider $\||\|\tilde{p}_{h,1}\|_{1,D_k}$. Since $\tilde{p}_{h,1}$ is discrete harmonic in D_k, we have

$$\|\tilde{p}_{h,1}\|_{1,D_k} \lesssim \|\tilde{p}_{h,1}\|_{\frac{1}{2},\partial D_k} \quad (k=2,3). \tag{3.33}$$

Define the interpolation operators $I^0_{\mathrm{F}_{1k}}$ and $I^0_{\partial\mathrm{F}_{1k}}$ as follows: for $\psi_h \in Z_h(G)$, the function $I^0_{\mathrm{F}_{1k}}\psi_h$ (resp. $I^0_{\partial\mathrm{F}_{1k}}\psi_h$) equals ψ_h at the nodes in the interior of F_{1k} (resp. on $\partial\mathrm{F}_{1k}$) and vanishes at all the other nodes (resp. at all the nodes not on $\partial\mathrm{F}_{1k}$). From the definition of $\tilde{p}_{h,1}$, we have $\tilde{p}_{h,1} = I^0_{\mathrm{F}_{1k}}\tilde{p}_{h,1} + I^0_{\partial\mathrm{F}_{1k}}\tilde{p}_{h,1}$ on ∂D_k. Then, it follows by (3.33) that

$$\begin{aligned}\|\tilde{p}_{h,1}\|_{1,D_k} &\lesssim \|I^0_{\mathrm{F}_{1k}}\tilde{p}_{h,1}\|_{\frac{1}{2},\partial D_k} + \|I^0_{\partial\mathrm{F}_{1k}}\tilde{p}_{h,1}\|_{\frac{1}{2},\partial D_k}\\ &\lesssim \|I^0_{\mathrm{F}_{1k}}\tilde{p}_{h,1}\|_{H^{\frac{1}{2}}_{00}(\mathrm{F}_{1k})} + \|\tilde{p}_{h,1}\|_{0,\partial\mathrm{F}_{1k}}\\ &= \|p_{h,1}\|_{H^{\frac{1}{2}}_{00}(\mathrm{F}_{1k})} + \|p_{h,1}\|_{0,\partial\mathrm{F}_{1k}}.\end{aligned}$$

Therefore, by using the "face" lemma and "edge" lemma (cf.[30] and [31]), we further obtain

$$\|\tilde{p}_{h,1}\|_{1,D_k} \lesssim \log(1/h)\|p_{h,1}\|_{\frac{1}{2},\partial D_1} \lesssim \log(1/h)\|p_{h,1}\|_{1,D_1} \quad (k=2,3).$$

Then, as in the estimate for $\|\mathbf{w}_h\|_{1,G}$ (but (3.31) needs to be used), we get

$$\|p_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{v}_h\|_{0,G}.$$

Combining this with (3.32), gives the second inequality in (3.14). Moreover, we can similarly derive (3.15) by (3.18) and (3.23). ♯

Remark 3.2 *The construction of the decomposition (3.24) is a bit technical. The main difficulty comes from the definition of the vector-valued function* $\mathbf{w}_h$*, which must satisfy* L^2 *stability. A natural idea is to extend* $\mathbf{w}_{h,1}$ *onto* G *such that the extension is discrete harmonic in* D_2 *and* D_3*, but the resulting extension may not satisfy the* L^2 *stability (3.31).*

4 Regular Helmholtz decompositions preserving local zero tangential complements

In this section we present regular Helmholtz decompositions for vector-valued functions that have zero tangential complements on some edges of a polyhedron. To this end, we need to build a serials of auxiliary results. Before giving these auxiliary results, we introduce a discrete **curl**-harmonic extension and describe its stability.

The discrete **curl**-harmonic extension operator $\mathbf{E}_h : V_h(\partial G) \to V_h(G)$ is defined as follows: for any $\Phi \in V_h(\partial G)$, $\mathbf{E}_h\Phi \in V_h(G)$ satisfies $\mathbf{E}_h\Phi \times \mathbf{n} = \Phi$ on ∂G, and

$$(\mathbf{curl}\,\mathbf{E}_h\Phi, \mathbf{curl}\,\mathbf{v}_h) = 0, \quad \forall\, \mathbf{v}_h \in V_h^0(G)$$

and

$$(\mathbf{E}_h\Phi,\ \nabla q_h) = 0, \ \ \forall q_h \in Z_h^0(G).$$

Let div_τ be the tangential divergence defined in [1], which was called surface *curl* in [29]. For $\mathbf{v}_h \in V_h(G)$, we have $curl_S\mathbf{v}_h = \mathrm{div}_\tau(\mathbf{n} \times \mathbf{v}_h) = (\mathbf{curl}\ \mathbf{v}_h) \cdot \mathbf{n}$; see [1]. For ease of understanding, we directly use the notation $(\mathbf{curl}\ \mathbf{v}_h) \cdot \mathbf{n}$ in the rest of this paper.

The following result can be found in [1].

Proposition 4.1 (trace inequality of vector-valued function) For any $\mathbf{v}_h \in V_h(G)$, we have

$$\|(\mathbf{curl}\ \mathbf{v}_h) \cdot \mathbf{n}\|_{-\frac{1}{2},\partial G} \lesssim \|\mathbf{curl}\ \mathbf{v}_h\|_{0,\ G}. \tag{4.1}$$

♯

Proposition 4.2 (stability of the **curl**-harmonic extension) For any $\mathbf{v}_h \in V_h(G)$, we have

$$\|\mathbf{curl}\ (\mathbf{E}_h(\mathbf{v}_h \times \mathbf{n})|_{\partial G})\|_{0,G} \lesssim \|(\mathbf{curl}\ \mathbf{v}_h) \cdot \mathbf{n}\|_{-\frac{1}{2},\partial G}. \tag{4.2}$$

Proof. Set $\Phi = (\mathbf{v}_h \times \mathbf{n})|_{\partial G}$. Let $q(\Phi) \in H(\mathbf{curl};\ G)$ be defined by (4.22)-(4.23) in [19] (see also [1]), and define $\tilde{\mathbf{w}}(\Phi) = \mathbf{curl}\ q(\Phi)$. Then, by (4.22)-(4.23) in [19], we have $\tilde{\mathbf{w}}(\Phi)\times\mathbf{n} = \Phi$ and we can verify that (notice that $div\ q(\Phi) = 0$ from Lemma 3.5 in [1])

$$\|\mathbf{curl}\tilde{\mathbf{w}}(\Phi)\|_{\delta,G} \lesssim \|div_\tau\Phi\|_{\delta-\frac{1}{2},\partial G}, \quad \forall\delta \in [0, \delta_0] \ \text{ for some } \delta_0 \in (\frac{1}{2}, 1).$$

Define $\tilde{\mathbf{w}}_h(\Phi) = \mathbf{r}_h\tilde{\mathbf{w}}(\Phi)$. By the interpolation estimate in [2, 7] and the inverse estimate in [2], we can further show that

$$\|\mathbf{curl}\ \tilde{\mathbf{w}}_h(\Phi)\|_{0,G} \lesssim \|div_\tau\Phi\|_{-\frac{1}{2},\partial G}.$$

Now the inequality (4.2) is a direct result of the above estimate and the minimal property of energy of $\mathbf{E}_h(\Phi)$. ♯

Lemma 4.1 *Let* F *be a (closed) face of* G*, and assume that* $\mathbf{v}_h \in V_h(G)$ *satisfies* $\mathbf{v}_h\cdot\mathbf{t}_{\partial\mathrm{F}} = 0$ *on* ∂F*. Then there exist* $p_h \in Z_h(G)$*,* $\mathbf{w}_h \in (Z_h(G))^3$ *and* $\mathbf{R}_h \in V_h(G)$*, which satisfy* $p_h = 0$*,* $\mathbf{w}_h = \mathbf{0}$ *and* $\mathbf{R}_h \cdot \mathbf{t}_{\partial\mathrm{F}} = 0$ *on* ∂F*, such that*

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h\mathbf{w}_h + \mathbf{R}_h, \tag{4.3}$$

with the following estimates

$$\|\mathbf{w}_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}, \quad \|\mathbf{w}_h\|_{0,G} + \|p_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{v}\|_{0,G} \tag{4.4}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{0,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}. \tag{4.5}$$

The conclusion is also valid for the case when F *is replaced by a union of some faces.*

Proof. We separate the proof into two steps.

Step 1: Establish the desired decomposition.

Let $\mathbf{v}_{h,\mathrm{F}} \in V_h(G)$ be an extension of $(\mathbf{v}_h \times \mathbf{n})|_{\mathrm{F}}$ into $V_h(G)$, such that $\lambda_e(\mathbf{v}_{h,\mathrm{F}}) = 0$ for any $e \subset \mathrm{F}_c = (\partial G \backslash \mathrm{F}) \cup \partial \mathrm{F}$, and it is discrete **curl**-harmonic on G. It follows by Lemma 3.1 that $\mathbf{v}_{h,\mathrm{F}}$ admits the decomposition

$$\mathbf{v}_{h,\mathrm{F}} = \mathbf{r}_h \Phi_{h,\mathrm{F}} + \nabla p_{h,\mathrm{F}} + \mathbf{R}_{h,\mathrm{F}} \tag{4.6}$$

with $\Phi_{h,\mathrm{F}} \in (Z_h(G) \cap H^1_{\mathrm{F}_c}(G))^3$, $p_{h,\mathrm{F}} \in Z_h(G) \cap H^1_{\mathrm{F}_c}(G)$ and $\mathbf{R}_{h,\mathrm{F}} \in V_h(G)$. Moreover, they satisfy

$$\|\Phi_{h,\mathrm{F}}\|_{1,G} \lesssim \|\mathbf{curl}\, \mathbf{v}_{h,\mathrm{F}}\|_{0,G}, \quad \|\Phi_{h,\mathrm{F}}\|_{0,G} + \|p_{h,\mathrm{F}}\|_{1,G} \lesssim \log(1/h) \|\mathbf{v}_{h,\mathrm{F}}\|_{0,G} \tag{4.7}$$

and

$$h^{-1} \|\mathbf{R}_{h,\mathrm{F}}\|_{0,G} \lesssim \|\mathbf{curl}\, \mathbf{v}_{h,\mathrm{F}}\|_{0,G}. \tag{4.8}$$

Then we define

$$\hat{\mathbf{v}}_{h,\mathrm{F}} = \mathbf{v}_h - (\nabla p_{h,\mathrm{F}} + \mathbf{r}_h \Phi_{h,\mathrm{F}} + \mathbf{R}_{h,\mathrm{F}}). \tag{4.9}$$

We can check that $\hat{\mathbf{v}}_{h,\mathrm{F}} \times \mathbf{n} = \mathbf{0}$ on F. By Lemma 3.1, the function $\hat{\mathbf{v}}_{\mathrm{F},h}$ has the decomposition

$$\hat{\mathbf{v}}_{h,\mathrm{F}} = \nabla \hat{p}_h + \mathbf{r}_h \hat{\Phi}_h + \hat{\mathbf{R}}_h \tag{4.10}$$

for some $\hat{p}_h \in Z_h(G) \cap H^1_{\mathrm{F}}(G)$, $\hat{\Phi}_h \in (Z_h(G) \cap H^1_{\mathrm{F}}(G))^3$ and $\hat{\mathbf{R}}_h \in V_h(G)$, such that

$$\|\hat{\Phi}_h\|_{1,G} \lesssim \|\mathbf{curl}\, \hat{\mathbf{v}}_{h,\mathrm{F}}\|_{0,G}, \quad \|\hat{\Phi}_h\|_{0,G} + \|\hat{p}_h\|_{1,G} \lesssim \log(1/h) \|\hat{\mathbf{v}}_{h,\mathrm{F}}\|_{0,G} \tag{4.11}$$

and

$$h^{-1} \|\hat{\mathbf{R}}_h\|_{0,G} \lesssim \|\mathbf{curl}\, \hat{\mathbf{v}}_{h,\mathrm{F}}\|_{0,G}. \tag{4.12}$$

Define

$$\mathbf{w}_h = \Phi_{h,\mathrm{F}} + \hat{\Phi}_h,\ \ p_h = p_{h,\mathrm{F}} + \hat{p}_h \ \text{ and }\ \mathbf{R}_h = \mathbf{R}_{h,\mathrm{F}} + \hat{\mathbf{R}}_h.$$

Then we get the decomposition

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h. \tag{4.13}$$

It is easy to see that p_h and $\mathbf{w}_h$ vanish on ∂F, and so $\mathbf{R}_h \cdot \mathbf{t}_{\partial \mathrm{F}} = 0$ on ∂F.

Step 2: Verify the desired estimate (4.4) for the decomposition (4.13).

By the definition of $\mathbf{w}_h$ and the triangle inequality, we have

$$\|\mathbf{w}_h\|_{1,G} \lesssim \|\Phi_{h,\mathrm{F}}\|_{1,G} + \|\hat{\Phi}_h\|_{1,G}.$$

This, along with (4.7), (4.11) and (4.10), leads to

$$\begin{aligned} \|\mathbf{w}_h\|_{1,\ G} &\lesssim \|\mathbf{curl}\, \mathbf{v}_{h,\mathrm{F}}\|_{0,G} + \|\mathbf{curl}\, \hat{\mathbf{v}}_{h,\mathrm{F}}\|_{0,G} \\ &\lesssim \|\mathbf{curl}\, \mathbf{v}_{h,\mathrm{F}}\|_{0,G} + \|\mathbf{curl}\, (\mathbf{r}_h \hat{\Phi}_h)\|_{0,G} + \|\mathbf{curl}\, \hat{\mathbf{R}}_h\|_{0,G}. \end{aligned}$$

Then, from (4.11), (4.12) and (4.9), together with (4.7) and (4.8), we further get that

$$\|\mathbf{w}_h\|_{1,\ G} \lesssim \|\mathbf{curl}\, \mathbf{v}_{h,\mathrm{F}}\|_{0,G} + \|\mathbf{curl}\, \mathbf{v}_h\|_{0,G}. \tag{4.14}$$

Therefore, by the definition of $\mathbf{v}_{h,\mathrm{F}}$ and using the stability of **curl**-harmonic extension, we have

$$\begin{aligned}\|\mathbf{w}_h\|_{1,\ G} &\lesssim \|(\mathbf{curl}\ \mathbf{v}_{h,\mathrm{F}})\cdot\mathbf{n}\|_{-\frac{1}{2},\partial G}+\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}\\ &\lesssim \|(\mathbf{curl}\ \mathbf{v}_h)\cdot\mathbf{n}\|_{-\frac{1}{2},\mathrm{F}}+\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}.\end{aligned} \tag{4.15}$$

On the other hand, using the known face $H^{-\frac{1}{2}}$-extension (cf. [16],[18] and [29]) and the trace inequality, we obtain

$$\|(\mathbf{curl}\ \mathbf{v}_h)\cdot\mathbf{n}\|_{-\frac{1}{2},\mathrm{F}} \lesssim \log(1/h)\|(\mathbf{curl}\ \mathbf{v}_h)\cdot\mathbf{n}\|_{-\frac{1}{2},\partial G}\lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}.$$

Substituting this into (4.15), yields the first inequality of (4.4). The second inequality in (4.4) and the inequality (4.5) can be derived similarly. ♯

From the above proof, we can obtain the following result

Corollary 4.1. Let F be a (closed) face of G, and Γ be a union of several faces of G. Assume that $\mathbf{v}_h\in V_h(G)$ satisfies $\mathbf{v}_h\cdot\mathbf{t}_{\partial\mathrm{F}}=0$ on ∂F and $\mathbf{v}_h\times\mathbf{n}=\mathbf{0}$ on Γ. Then there exist $p_h\in Z_h(G)$, $\mathbf{w}_h\in(Z_h(G))^3$ and $\mathbf{R}_h\in V_h(G)$, which satisfy $p_h=0$, $\mathbf{w}_h=\mathbf{0}$ on $\Gamma\cup\partial$F and $\lambda_e(\mathbf{R}_h)=0$ for any $e\subset\Gamma\cup\partial$F, such that

$$\mathbf{v}_h=\nabla p_h+\mathbf{r}_h\mathbf{w}_h+\mathbf{R}_h. \tag{4.16}$$

Moreover, we have the following estimates

$$\|\mathbf{w}_h\|_{1,G}\lesssim\log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G},\quad \|\mathbf{w}_h\|_{0,G}+\|p_h\|_{1,G}\lesssim\log(1/h)\|\mathbf{v}\|_{0,G} \tag{4.17}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{0,G}\lesssim\log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}. \tag{4.18}$$

Proof. As in the proof of Lemma 4.1, we set $\mathrm{F}_c=(\partial G\backslash\mathrm{F})\cup\partial\mathrm{F}$ and use Lemma 3.1 for F^c and $\Gamma\cup$ F, respectively. ♯

Lemma 4.2 *Let* E *be a (closed) edge of* G*, and* $\mathbf{v}_h$ *be a finite element function in* $V_h(G)$ *such that* $\mathbf{v}_h\cdot\mathbf{t}_{\mathrm{E}}=0$ *on* E*. Then* $\mathbf{v}_h$ *admits a decomposition*

$$\mathbf{v}_h=\nabla p_h+\mathbf{r}_h\mathbf{w}_h+\mathbf{R}_h$$

for some $p_h\in Z_h(G)$*,* $\mathbf{w}_h\in(Z_h(G))^3$ *and* $\mathbf{R}_h\in V_h(G)$*, which satisfy* $p_h=0$*,* $\mathbf{w}_h=\mathbf{0}$*,* $\mathbf{R}_h\cdot\mathbf{t}_{\mathrm{E}}=0$ *on* E*. Moreover, the following estimates hold*

$$\|\mathbf{w}_h\|_{1,G}\lesssim\log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,G},\quad \|p_h\|_{1,G}\lesssim\log(1/h)\|\mathbf{v}\|_{\mathbf{curl},G} \tag{4.19}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{0,G}\lesssim\log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}. \tag{4.20}$$

The conclusion is also valid for the case when E *is replaced by a connected union of several edges on one face of* G*.*

Proof. We separate the proof into three steps.

Step 1: Establish an edge-related decomposition.

Let F be a face containing the edge E. We first consider a decomposition of the tangential component $\mathbf{v}_h\cdot\mathbf{t}_{\partial\mathrm{F}}$ of $\mathbf{v}_h$ on ∂F. For convenience, we write $\mathrm{E}^c=\partial\mathrm{F}\backslash\mathrm{E}$. Let s be the arc-length along E^c, taking values from 0 to l_0, where l_0 is the total length of E^c. In terms of

s, the function $\mathbf{v}_h \cdot \mathbf{t}_{\mathrm{E}^c}$ is piecewise linear on the interval $[0, l_0]$, denoted by $\hat{v}(s)$. Then we define

$$C_{\mathrm{E}} = \frac{1}{l_0}\int_0^{l_0} \hat{v}(s)\, ds, \quad \phi_{\mathrm{E}}(t) = \int_0^t (\hat{v}(s) - C_{\mathrm{E}}) ds\,, \quad \forall\, t \in [0, l_0]\,.$$

Clearly we see $\phi_{\mathrm{E}}(t)$ vanishes at $t = 0$ and l_0. We can extend ϕ_{E} naturally by zero onto E, then extend by zero into ∂G and G such that its extensions $\tilde{\phi}_{\mathrm{E}} \in Z_h(G)$. In the following, we define an extension $\tilde{C}_{\mathrm{E}}$ of C_{E} such that $\tilde{C}_{\mathrm{E}}$ belongs to $(Z_h(G))^3$ and vanishes on E. Moreover, we require that $\tilde{C}_{\mathrm{E}}$ satisfies $(\mathbf{r}_h \tilde{C}_{\mathrm{E}}) \cdot \mathbf{t}_{\partial \mathrm{F}} = C_{\mathrm{E}}$ on $\mathrm{E}^c = \partial \mathrm{F} \backslash \mathrm{E}$ and

$$\|\tilde{C}_{\mathrm{E}}\|_{1,\hat{\Omega}} \lesssim |C_{\mathrm{E}}|. \tag{4.21}$$

Let Ξ denote the set of the nodes on G, and let $\Im$ denote the set of the nodes in E^c. Then the values of the vector-valued function $\tilde{C}_{\mathrm{E}}$ on the nodes in $G \backslash \Im$ are defined to be zero. Moreover, the values of the vector-valued function $\tilde{C}_{\mathrm{E}}$ on the nodes in $\Im$ are defined such that $\tilde{C}_{\mathrm{E}}$ is linear on each (coarse) edge on E^c and $\|\tilde{C}_{\mathrm{E}}\|^2_{0,\mathrm{E}^c}$ reaches the minimal value under the constraint $(\mathbf{r}_h \tilde{C}_{\mathrm{E}}) \cdot \mathbf{t}_{\partial \mathrm{F}} = C_{\mathrm{E}}$ on E^c. Notice that the number of degrees of freedom of the function $\tilde{C}_{\mathrm{E}}$, which equals three times the number of vertices in E^c, is greater than the number of coarse edges contained in E^c. Then the minimization problem (with a quadric subject functional and compatible linear constraints) has a solution. In particular, if $C_{\mathrm{E}} = 0$, the desired vector-valued function $\tilde{C}_{\mathrm{E}} = \mathbf{0}$. Since each edge on E^c is of size $O(1)$, we have

$$\|\tilde{C}_{\mathrm{E}}\|_{0,\mathrm{E}^c} \lesssim |C_{\mathrm{E}}|.$$

Moreover, by the definition of $\tilde{C}_{\mathrm{E}}$ and the discrete norms, we get

$$\|\tilde{C}_{\mathrm{E}}\|_{1,\hat{\Omega}} \lesssim \|\tilde{C}_{\mathrm{E}}\|_{0,\mathrm{E}^c}.$$

Thus the inequality (4.21) indeed holds. By the definitions of $\tilde{\phi}_{\mathrm{E}}$ and $\tilde{C}_{\mathrm{E}}$, one can verify that (cf. [29]) that (since $\mathbf{v}_h \cdot \mathbf{t}_{\mathrm{E}} = 0$)

$$\mathbf{v}_h \cdot \mathbf{t}_{\partial \mathrm{F}} = (\nabla \tilde{\phi}_{\mathrm{E}}) \cdot \mathbf{t}_{\partial \mathrm{F}} + (\mathbf{r}_h \tilde{C}_{\mathrm{E}}) \cdot \mathbf{t}_{\partial \mathrm{F}} \tag{4.22}$$

and

$$\|\tilde{\phi}_{\mathrm{E}}\|_{1,G} \lesssim \log(1/h)(\|\mathbf{v}_h\|_{0,G} + \|\mathbf{curl}\, \mathbf{v}_h\|_{0,G}). \tag{4.23}$$

Step 2: Construct the desired decomposition in Lemma 4.2. For the purpose, we set

$$\hat{\mathbf{v}}_{h,\mathrm{E}} = \mathbf{v}_h - (\nabla \tilde{\phi}_{\mathrm{E}} + \mathbf{r}_h \tilde{C}_{\mathrm{E}}). \tag{4.24}$$

By (4.22) we know $\hat{\mathbf{v}}_{h,\mathrm{E}} \cdot \mathbf{t}_{\partial \mathrm{F}} = 0$ on ∂F. For the function $\hat{\mathbf{v}}_{h,\mathrm{E}}$ in (4.24), by Lemma 4.1 one can find functions $\hat{p}_h \in Z_h(G)$, $\hat{\mathbf{w}}_h \in (Z_h(G))^3$ and $\hat{\mathbf{R}}_h \in V_h(G)$ such that $\hat{p}_h$ and $\hat{\mathbf{w}}_h$ vanish on ∂F, and

$$\hat{\mathbf{v}}_{h,\mathrm{E}} = \nabla \hat{p}_h + \mathbf{r}_h \hat{\mathbf{w}}_h + \hat{\mathbf{R}}_h,$$

with the following estimates

$$\|\hat{\mathbf{w}}_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{curl}\, \hat{\mathbf{v}}_{h,\mathrm{E}}\|_{0,G}, \quad \|\hat{p}_h\|_{1,G} \lesssim \log(1/h)\|\hat{\mathbf{v}}_{h,\mathrm{E}}\|_{0,G} \tag{4.25}$$

and

$$h^{-1}\|\hat{\mathbf{R}}_h\|_{0,\hat{\Omega}} \lesssim \log(1/h)\|\mathbf{curl}\, \hat{\mathbf{v}}_{h,\mathrm{E}}\|_{0,\hat{\Omega}}. \tag{4.26}$$

Now by defining

$$p_h = \tilde{\phi}_{\rm E} + \hat{p}_h, \quad \mathbf{w}_h = \tilde{C}_{\rm E} + \hat{\mathbf{w}}_h \ \text{ and } \ \mathbf{R}_h = \hat{\mathbf{R}}_h\,,$$

we get the final decomposition

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h \tag{4.27}$$

such that $p_h = 0$ and $\mathbf{w}_h = 0$ on E.

Step 3: Derive the desired estimate in Lemma 4.2 for the decomposition (4.27).

Noting that $\mathbf{v}_h \cdot \mathbf{t}_{\rm E} = 0$ on E, so $\mathbf{v}_h \cdot \mathbf{t}_{\partial \rm F} = 0$ on E, we have by the Green's formula on F and change of variables (cf. [29]) that (with l being the total arclength of ∂F)

$$C_{\rm E} = \frac{1}{l_0}\int_0^l \hat{v}(s)ds = \frac{1}{l_0}\int_{\rm F} (\mathbf{curl}\ \mathbf{v}_h)\cdot \mathbf{n} ds. \tag{4.28}$$

Using the face $H^{-1/2}$-extension (cf. [16],[18] and [29])) again, we have

$$\|(\mathbf{curl}\ \mathbf{v}_h)\cdot \mathbf{n}\|_{-\frac{1}{2},\rm F} \lesssim \log^{\frac{1}{2}}(1/h)\|(\mathbf{curl}\ \mathbf{v}_h)\cdot \mathbf{n}\|_{-\frac{1}{2},\,\partial G}$$

and further get by (4.1)

$$|\int_{\rm F} (\mathbf{curl}\ \mathbf{v}_h)\cdot \mathbf{n} ds| \lesssim \|(\mathbf{curl}\ \mathbf{v}_h)\cdot \mathbf{n}\|_{-\frac{1}{2},\rm F}\cdot \|1\|_{\frac{1}{2},\ \rm F} \lesssim \log^{\frac{1}{2}}(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,G}.$$

This, along with (4.28), leads to

$$|C_{\rm E}| \lesssim \log^{\frac{1}{2}}(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,G}.$$

Then, by (4.21) we obtain

$$\|\tilde{C}_{\rm E}\|_{1,\hat{\Omega}} \lesssim \log^{\frac{1}{2}}(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,G}.$$

By the definition of $\mathbf{w}_h$ and the above inequality, together with (4.25), (4.24) and (4.23), we deduce the first estimate in (4.19)

In a similar way, we can prove the second estimate in (4.19) and the inequality (4.20) by (4.26). ♯

Remark 4.1 *There is a key difference in the proof of the above lemma from that of Lemma 4.3 in [17]: since the extension $\tilde{C}_{\rm E}$ in the above proof must belong to the space $(Z_h(G))^3$, $\tilde{C}_{\rm E}$ can not be defined to be the natural zero extension of $C_{\rm E}$ as in [17]. The same problem will appear in the proofs of the lemmas below.*

Lemma 4.3 *Let* V *be a vertex of G and $\mathbf{v}_h$ be a function in $V_h(G)$. Then we can write $\mathbf{v}_h$ as*

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h$$

for some $p_h \in Z_h(G)$, $\mathbf{w}_h \in (Z_h(G))^3$ and $\mathbf{R}_h \in V_h(G)$ satisfying $p_h(\rm V) = 0$ and $\mathbf{w}_h(\rm V) = \mathbf{0}$. Moreover, we have

$$\|\mathbf{w}_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}, \quad \|\nabla p_h\|_{0,G} \lesssim \|\mathbf{v}_h\|_{\mathbf{curl},G} \tag{4.29}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{0,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}. \tag{4.30}$$

Proof. Consider a (closed) face F containing V as its vertex. Like Step 1 in the proof of Lemma 4.2, we can define $\phi_{\partial \mathrm{F}}$ to be a function that is piecewise linear and continuous on ∂F such that $\phi_{\partial \mathrm{F}}(\mathrm{V}) = 0$, and define $C_{\partial \mathrm{F}}$ to be a constant such that $\mathbf{v}_h \cdot \mathbf{t}_{\partial \mathrm{F}} = \phi'_{\partial \mathrm{F}} + C_{\partial \mathrm{F}}$ on ∂F. In fact, they can be defined as

$$C_{\partial \mathrm{F}} = \frac{1}{l}\int_0^l (\mathbf{v}_h \cdot \mathbf{t}_{\partial \mathrm{F}})(s)\,ds, \quad \phi_{\partial \mathrm{F}}(t) = \int_0^t (\mathbf{v}_h \cdot \mathbf{t}_{\partial \mathrm{F}} - C_{\partial \mathrm{F}})(s)ds\,, \quad \forall\, t \in [0, l]\,,$$

where l is the length of ∂F and $t = 0$ (and $t = l$) corresponds the vertex V. Let $c = \gamma_{\mathrm{E}}(\phi_{\partial \mathrm{F}})$ denote the average of $\phi_{\partial \mathrm{F}}$ on E, where E is an edge or a union of several edges of F. Define an extension $\tilde{\phi}_{\mathrm{V}} \in Z_h(G)$ of $\phi_{\partial \mathrm{F}}$, such that $\tilde{\phi}_{\mathrm{V}}$ equals to the average $c = \gamma_{\mathrm{E}}(\phi_{\partial \mathrm{F}})$ at all the nodes on G except those on ∂F. Then (cf. [29])

$$\|\nabla\tilde{\phi}_{\mathrm{V}}\|_{0,G} = \|\nabla(\tilde{\phi}_{\mathrm{V}} - c)\|_{0,G} \lesssim \|\phi_{\partial \mathrm{F}} - c\|_{0,\partial \mathrm{F}} \lesssim \|\phi'_{\partial \mathrm{F}}\|_{H^{-1}(\partial \mathrm{F})} \lesssim \log(1/h)\|\mathbf{v}_h\|_{\mathbf{curl},\, G}. \quad (4.31)$$

We define a similar extension $\tilde{C}_{\mathrm{V}}$ of $C_{\partial \mathrm{F}}$ with $\tilde{C}_{\mathrm{E}}$ (defined in Lemma 4.2), such that $\tilde{C}_{\mathrm{V}}$ belongs to $(Z_h(G))^3$ and vanishes at V, and it satisfies the condition $(\mathbf{r}_h\tilde{C}_{\mathrm{V}}) \cdot \mathbf{t}_{\partial \mathrm{F}} = C_{\partial \mathrm{F}}$ on ∂F and the stability

$$\|\tilde{C}_{\mathrm{V}}\|_{1,G} \lesssim \|\tilde{C}_{\mathrm{V}}\|_{0,\partial \mathrm{F}} \lesssim |C_{\partial \mathrm{F}}| \lesssim \log^{\frac{1}{2}}(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,G}.$$

Define

$$\hat{\mathbf{v}}_{h,\mathrm{V}} = \mathbf{v}_h - (\tilde{\phi}_{\mathrm{V}} + \mathbf{r}_h\tilde{C}_{\mathrm{V}}).$$

Then we have $\hat{\mathbf{v}}_{h,\mathrm{V}} \cdot \mathbf{t}_{\partial \mathrm{F}} = 0$. As in the proof of Lemma 4.2, we can use Lemma 4.1 for $\hat{\mathbf{v}}_{h,\mathrm{V}}$ to build the desired decomposition of $\mathbf{v}_h$. ♯

Remark 4.2 *Comparing the second inequality in (4.19), we find that the second inequality in (4.29) holds only for the semi-norm of p_h. The main reason is that a stable estimate of $\|\tilde{\phi}_{\mathrm{V}}\|_{0,G}$ can not be built except that the constant c in (4.31) vanishes.*

Lemma 4.4 *Let Γ be a (closed) union of some faces of G, and E be a closed edge of G with $\mathrm{E} \not\subseteq \Gamma$. Assume that $\mathbf{v}_h \in V_h(G)$ satisfies $\mathbf{v}_h \times \mathbf{n} = 0$ on Γ and $\mathbf{v}_h \cdot \mathbf{t}_{\mathrm{E}} = 0$ on E. Then $\mathbf{v}_h$ can be decomposed as*

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h\mathbf{w}_h + \mathbf{R}_h \quad (4.32)$$

for some $p_h \in Z_h(G)$ and $\mathbf{w}_h \in (Z_h(G))^3$ and $\mathbf{R}_h \in V_h(G)$ such that p_h and $\mathbf{w}_h$ vanish on $\Gamma \cup \mathrm{E}$. Moreover, we have

$$\|\mathbf{w}_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}, \quad \|p_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{v}_h\|_{\mathbf{curl},G} \quad (4.33)$$

and

$$h^{-1}\|\mathbf{R}_h\|_{0,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}. \quad (4.34)$$

The result is also valid when E is replaced by a connected union of several edges on one face of G.

Proof. The position relations between E and Γ have two possible situations: (i) there exists a face F containing E such that $\mathrm{E} \cup (\mathrm{F} \cap \Gamma)$ is a connected set in ∂F (which includes three cases: (a) $\mathrm{F} \cap \Gamma = \emptyset$, (b) $\mathrm{F} \cap \Gamma$ is an endpoint of E, (c) $\mathrm{F} \cap \Gamma$ is just an edge E$'$ adjoining E); (ii) $\mathrm{E} \cap \Gamma = \emptyset$ and the intersection of any face containing E with Γ is an edge E' that does not adjoin with E, i.e., $\mathrm{E} \cup \mathrm{E}'$ is not a connected set in the boundary of this face. For the situation (i), the results in Lemma 4.4 can be built as in the proof of Lemma 4.2 (for the case (c), we replace E by $E \cup E'$ since the function $\mathbf{v}_h$ has the zero degrees of freedom on $E \cup E'$), but using **Corollary 4.1** instead of Lemma 3.1. Now we consider the situation (ii) (see Figure 2). In this situation the lemma can not be proved as in Lemma 4.2 since the defining function $\tilde{\phi}_E$ may not vanish on Γ, so we have to combine the ideas in the proofs of Lemma 3.1, Lemma 4.1 and Lemma 4.2.

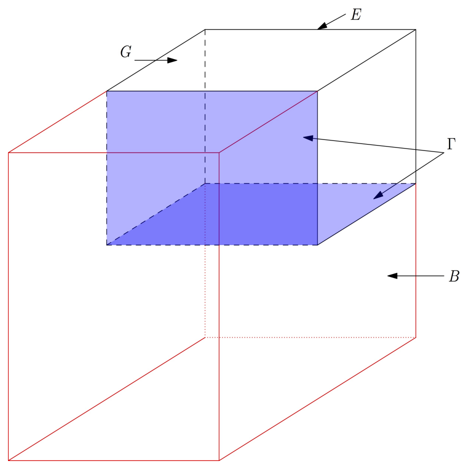

Figure 2: The small cube is G; the shaded part denotes Γ; the large cuboid is the domain B

As in the proof of Lemma 3.1, let B be a polyhedron domain containing G as its subdomain (when G is convex, we can require that B is also convex) such that $\partial G \cap \partial B = \partial G \backslash \Gamma$ and the size of the complement $D = B \backslash G$ is a positive number independent of h. It is easy to see that $\partial G \cap \partial D = \Gamma$. We extend $\mathbf{v}_h$ onto the global B by zero, i.e., the extension $\tilde{\mathbf{v}}_h$ satisfying $\tilde{\mathbf{v}}_h = \mathbf{v}_h$ on G and $\tilde{\mathbf{v}}_h = \mathbf{0}$ on $\bar{D}$. Since $\mathbf{v}_h \times \mathbf{n} = \mathbf{0}$ on Γ, we have $\tilde{\mathbf{v}}_h \in H(\mathbf{curl};\ B)$. Of course, E is also an edge of the auxiliary polyhedron B.

We can construct an auxiliary grids in D, then we obtain a partition on the global domain B. Let $V_h(B)$ be the resulting edge finite element space on B. Then $\tilde{\mathbf{v}}_h \in V_h(B)$. We choose a (closed) face F of B such that F contains E as its edge, and set $F_c = (\partial B \backslash F) \cup \partial F$. Let $\tilde{\phi}_E \in Z_h(B)$ and $\tilde{C}_E \in (Z_h(B))^3$ be the functions defined as in Lemma 4.2. Then $\tilde{\phi}_E$ and $\tilde{C}_E$ vanish on E, and $(\tilde{\mathbf{v}}_h - \tilde{C}_E - \nabla\tilde{\phi}_E) \cdot \mathbf{t}_{\partial F} = 0$ on ∂F. As in Lemma 4.1, but using the continuous results (3.1) and (3.2), we can build a decomposition

$$\tilde{\mathbf{v}}_h - \tilde{C}_E - \nabla\tilde{\phi}_E = \nabla(p_F + p_{F_c}) + (\mathbf{w}_F + \mathbf{w}_{F_c}) \quad \text{on} \quad B,$$

with $p_F \in H^1_F(B)$, $p_{F_c} \in H^1_{F_c}(B)$, $\mathbf{w}_F \in (H^1_F(B))^3$ and $\mathbf{w}_{F_c} \in (H^1_{F_c}(B))^3$. Namely,

$$\tilde{\mathbf{v}}_h = \nabla(\tilde{\phi}_E + p_F + p_{F_c}) + (\tilde{C}_E + \mathbf{w}_F + \mathbf{w}_{F_c}) \quad \text{on} \quad B. \tag{4.35}$$

Moreover, the following inequalities hold

$$\|\tilde{C}_E\|_{1,B} + \|\mathbf{w}_F\|_{1,B} + \|\mathbf{w}_{F_c}\|_{1,B} \lesssim \log(1/h)\|\mathbf{curl}\,\tilde{\mathbf{v}}_h\|_{0,B} \tag{4.36}$$

and

$$\|\tilde{\phi}_E\|_{1,B} + \|p_F\|_{1,B} + \|p_{F_c}\|_{1,B} \lesssim \log(1/h)\|\tilde{\mathbf{v}}_h\|_{\mathbf{curl},B}. \tag{4.37}$$

Define $\varphi = \tilde{\phi}_E + p_F + p_{F_c}$ and $\mathbf{w} = \tilde{C}_E + \mathbf{w}_F + \mathbf{w}_{F_c}$. By (4.35), we have

$$\tilde{\mathbf{v}}_h = \nabla\varphi + \mathbf{w} \quad \text{on} \quad B. \tag{4.38}$$

Then $\varphi|_D \in H^2(D)$ since $\mathbf{w} \in (H^1(B))^3$ and $\tilde{\mathbf{v}}_h = \mathbf{0}$ on D. Let δ_0 be a positive constant independent of h. We choose a domain $D_G \subset G$ such that $\bar{D}_G \cap \bar{D} = \Gamma$ and the size of D_G is about δ_0. Notice that $\text{E} \cap \Gamma = \emptyset$, we can require $dist(\text{E}, D_G) \geq \delta_0$. As in the proof of Lemma 3.1, we can define a stable extension $\tilde{\varphi}$ of $\varphi|_D$ from D onto the global B such that $\tilde{\varphi} \in H^2(B)$ and vanishes on $G \backslash D_G$.

Set $p = (\varphi - \tilde{\varphi})|_G$ and $\Phi = (\mathbf{w} + \nabla\tilde{\varphi})|_G$. Then $p \in H^1_\Gamma(G)$ and $\Phi \in (H^1_\Gamma(G))^3$. It follows by (4.38) that

$$\mathbf{v}_h = \tilde{\mathbf{v}}_h = \nabla\varphi + \mathbf{w} = \nabla p + \Phi \quad \text{ on } G.$$

Then there is a function $p_h \in Z_h(G)$ such that

$$\mathbf{v}_h = \mathbf{r}_h(\nabla p + \Phi) = \nabla p_h + \mathbf{r}_h\Phi \quad \text{ on } G. \tag{4.39}$$

Since p vanishes on Γ, the function p_h also vanishes on Γ. On the other hand, by the definitions of p and φ we have

$$\mathbf{r}_h\nabla p = \mathbf{r}_h\nabla\tilde{\phi}_\text{E} + \mathbf{r}_h\nabla p_\text{F} + \mathbf{r}_h\nabla p_{\text{F}_c} - \mathbf{r}_h\nabla\tilde{\varphi} = \nabla\tilde{\phi}_\text{E} + \nabla p_{h,\text{F}} + \nabla p_{h,\text{F}_c} - \nabla\tilde{\varphi}_h.$$

Moreover, from the definitions of $\tilde{\phi}_\text{E}$, p_F, p_{F_c} and $\tilde{\varphi}$, we know that $\tilde{\phi}_\text{E}$, $p_{h,\text{F}}$, p_{h,F_c} and $\tilde{\varphi}_h$ vanish at E (since $dist(\text{E}, D_G) \geq \delta_0$). Thus $p_h = \tilde{\phi}_\text{E} + p_{h,\text{F}} + p_{h,\text{F}_c} - \tilde{\varphi}_h$ vanishes at E. Let $\Pi_h : (H^1(G))^3 \to (Z_h(G))^3$ denote the Scott-Zhang interpolation operator, and define $\mathbf{w}_h = \Pi_h\Phi$ and $\mathbf{R}_h = \mathbf{r}_h(I - \Pi_h)\Phi$. Similarly, we can show that $\mathbf{w}_h$ vanishes on Γ and E. Then the decomposition (4.32) follows by (4.39).

The estimates in (4.33) can be obtained by using (4.36)-(4.37), the stability of the extension $\tilde{\varphi}$ and the approximation of Π_h (refer to the proof of Lemma 3.1). Here we need to use the fact that the stability constant of the extension $\tilde{\varphi}$ is independent of h (since the size δ_0 of D_G is independent of h). ♯

We point out that, if the edge E in Lemma 4.4 is replaced by a vertex V (refer to Lemma 4.3), we fail to obtain a similar result with Lemma 4.4. The difficulty comes from the fact that the estimate (4.37) does not hold yet if replacing $\tilde{\phi}_\text{E}$ by $\tilde{\phi}_\text{V}$ (see Remark 4.2), so stability estimates of the extension $\tilde{\varphi}$ can not be built. Because of this, we have to take special care for the case with a vertex.

Lemma 4.5 *Let Γ be a (closed) union of some faces of G, and V be a vertex of G (V $\notin \Gamma$). Assume that $\mathbf{v}_h \in V_h(G)$ satisfies $\mathbf{v}_h \times \mathbf{n} = 0$ on Γ. If there exist a face F containing V such that $\mathbf{v}_h$ satisfies $\gamma_\text{E}(\phi_{\partial\text{F}}) = 0$ for an edge E (or a union of several edges) of F , where the function $\phi_{\partial\text{F}}$ was defined in the proof of Lemma 4.3, then the decomposition and estimates in Lemma 4.4 still hold, with p_h and $\mathbf{w}_h$ vanishing on Γ and V. The results are also valid when Γ is replaced by a connected union of several edges on one face.*

Proof. We first consider the case with $\text{F} \cap \Gamma = \emptyset$. As in the proof of Lemma 4.3, we define $\tilde{\phi}_\text{V}$ and $\tilde{C}_\text{V}$ and set

$$\hat{\mathbf{v}}_{h,\text{V}} = \mathbf{v}_h - (\tilde{\phi}_\text{V} + \mathbf{r}_h\tilde{C}_\text{V}).$$

Then we have $\lambda_e(\hat{\mathbf{v}}_{h,\text{V}}) = 0$ for any $e \subset \partial\text{F} \cup \Gamma$ by the definitions of $\tilde{\phi}_\text{V}$ and $\tilde{C}_\text{V}$, together with the assumption $\gamma_\text{E}(\phi_{\partial\text{F}}) = 0$. Thus we can use **Corollary 4.1** for $\hat{\mathbf{v}}_{h,\text{V}}$ to build the desired decomposition of $\mathbf{v}_h$ and the estimates (refer to the proof of Lemma 4.2).

The case with $\text{F} \cap \Gamma \neq \emptyset$ is a bit complicated. Let B and $\tilde{\mathbf{v}}_h$ be defined in the proof of Lemma 4.4, and let $C_{\partial\text{F}}$ and $\phi_{\partial\text{F}}$ be defined in Lemma 4.3. We use $\tilde{\text{F}} \subset \partial B$ to denote a face of B such that $\tilde{\text{F}}$ contains F (see Figure 3). We need to define two functions $\tilde{\phi}_\text{V} \in Z_h(B)$ and $\tilde{C}_\text{V} \in (Z_h(B))^3$, which can be regarded as extensions of $\phi_{\partial\text{F}}$ and $C_{\partial\text{F}}$.

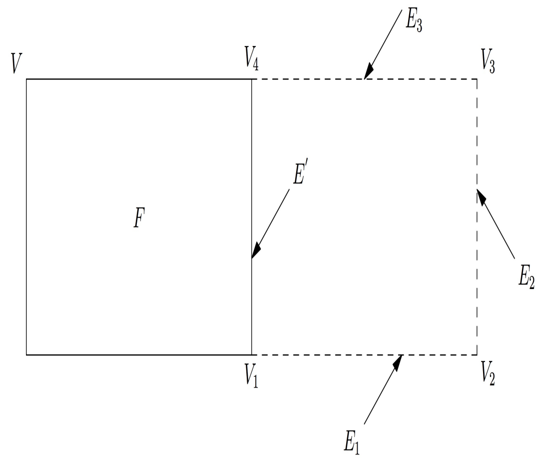

Figure 3: The left rectangle is F; the large rectangle denotes $\tilde{\text{F}}$.

At first we extend the definition of $\phi_{\partial\text{F}}$ onto $\partial\tilde{\text{F}}$. Without loss of generality, we assume that $\text{F}\cap\Gamma=\text{E}'$ be an edge of F, and $\tilde{\text{F}}\backslash\text{F}$ have four edges, one of which is just E' and the others of which are denoted by E_1, E_2 and E_3, where E_1 and E_3 are adjacent with E' but $\text{E}'\cap\text{E}_2=\emptyset$. Then $\partial\tilde{\text{F}}=(\partial\text{F}\backslash\text{E}')\cup\text{E}_1\cup\text{E}_2\cup\text{E}_3$. For convenience, we use $\text{V}_1,\text{V}_2,\text{V}_3$ and V_4 to denote the four vertices of $\tilde{\text{F}}\backslash\text{F}$, where V_1 and V_4 are two endpoints of E', V_2 and V_3 are two endpoints of E_2.

Let t_i denote the arc-length coordinate of V_i $(i=1,\cdots,4)$ with $t_1<t_2<t_3<t_4$. Define $\phi_{\partial\tilde{\text{F}}}\in Z_h(\partial\tilde{\text{F}})$ as follows: $\phi_{\partial\tilde{\text{F}}}=\phi_{\partial\text{F}}$ on $\partial\text{F}\backslash\text{E}'$ and $\phi_{\partial\tilde{\text{F}}}$ is linear on E_i $(i=1,2,3)$ with $\phi_{\partial\tilde{\text{F}}}(t_1)=\phi_{\partial\tilde{\text{F}}}(t_2)=\phi_{\partial\text{F}}(t_1)$ and $\phi_{\partial\tilde{\text{F}}}(t_3)=\phi_{\partial\tilde{\text{F}}}(t_4)=\phi_{\partial\text{F}}(t_4)$. Then $\tilde{\phi}_{\text{V}}\in Z_h(B)$ is defined such that $\tilde{\phi}_{\text{V}}=\phi_{\partial\tilde{\text{F}}}$ on $\partial\tilde{\text{F}}$ and it vanishes at all the nodes except on $\partial\tilde{\text{F}}$. As in the proof of Lemma 4.2, we define $\tilde{C}_{\text{V}}\in(Z_h(B))^3$ such that: (i) $\tilde{C}_{\text{V}}=\mathbf{0}$ at V; (ii) $\|\tilde{C}_{\text{V}}\|_{0,\partial\tilde{\text{F}}}$ reaches the minimal value under the constraints $(\mathbf{r}_h\tilde{C}_{\text{V}})\cdot\mathbf{t}_{\partial\tilde{\text{F}}}=C_{\partial\text{F}}$ on $\partial\text{F}\backslash\text{E}'$ and E_2, $(\mathbf{r}_h\tilde{C}_{\text{V}})\cdot\mathbf{t}_{\partial\tilde{\text{F}}}=0$ on E_1 and E_3; (iii) $\tilde{C}_{\text{V}}$ vanishes at all the nodes except on $\partial\tilde{\text{F}}$. Since $\mathbf{v}_h\cdot\mathbf{t}_{\text{E}}=0$, $\phi_{\partial\text{F}}$ is also linear on E' and $\phi_{\partial\text{F}}(t_4)-\phi_{\partial\text{F}}(t_1)=-C_{\partial\text{F}}$. Thus, by the definition of $\phi_{\partial\tilde{\text{F}}}$, we have $\phi'_{\partial\tilde{\text{F}}}=-C_{\partial\text{F}}$ on E_3 and $\phi'_{\partial\tilde{\text{F}}}=0$ on E_1 and E_2. Furthermore, we can verify that

$$(\tilde{\mathbf{v}}_h-\nabla\tilde{\phi}_{\text{V}}-\tilde{C}_{\text{V}})\cdot\mathbf{t}_{\partial\tilde{\text{F}}}=(\tilde{\mathbf{v}}_h-\tilde{C}_{\text{V}})\cdot\mathbf{t}_{\partial\tilde{\text{F}}}-\tilde{\phi}'_{\text{V}}=0\quad\text{on }\partial\tilde{\text{F}}.$$

Then the results can be built as in the proof of Lemma 4.4. To this end, we need to estimate $\|\tilde{C}_{\text{V}}\|_{1,B}$ and $\|\tilde{\phi}_{\text{V}}\|_{1,B}$ as in Lemma 4.3. Thanks to the assumption $\gamma_{\text{E}}(\phi_{\partial\text{F}})=0$, the L^2 norm $\|\tilde{\phi}_{\text{V}}\|_{0,B}$ is also bounded with a logarithmical factor only. ♯

Remark 4.3 *The condition $\gamma_{\text{E}}(\phi_{\partial\text{F}})=0$ in Lemma 4.5 seems absolutely necessary. In fact, we can construct a counterexample: define $\mathbf{v}_h=\nabla\phi_h$ with $\phi\in Z_h(G)$ and vanishing at all the nodes except* V*, and choose Γ as a union of all the faces that do not contain* V*. For this example, the decomposition satisfying the conditions in Lemma 4.5 does not exist since the estimates mean that $\mathbf{w}_h=\mathbf{R}_h=\mathbf{0}$ and so $\nabla p_h=\mathbf{v}_h=\nabla\phi_h$ by the decomposition, i.e., $p_h-\phi_h$=const, but the function $p_h-\phi_h$ must vanish on Γ and does not vanish at* V.

Lemma 4.6 *Let $\text{E}_1,\cdots,\text{E}_n$ be (closed) edges of G, which satisfy $\text{E}_l\cap\text{E}_j=\emptyset$ for any two different j and l. Assume that $\mathbf{v}_h\in V_h(G)$ satisfies $\mathbf{v}_h\cdot\mathbf{t}_{\text{E}_l}=0$ on each E_l. Then $\mathbf{v}_h$ can be decomposed as*

$$\mathbf{v}_h=\nabla p_h+\mathbf{r}_h\mathbf{w}_h+\mathbf{R}_h$$

for some $p_h \in Z_h(G)$ and $\mathbf{w}_h \in (Z_h(G))^3$ and $\mathbf{R}_h \in V_h(G)$ such that p_h and $\mathbf{w}_h$ vanish on each edge E_l. Moreover, we have

$$\|\mathbf{w}_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}, \quad \|p_h\|_{1,G} \lesssim \log(1/h)\mathbf{v}_h\|_{\mathbf{curl},G} \tag{4.40}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{0,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}. \tag{4.41}$$

The results are also valid when E_l is replaced by a connected union of edges on one face.

Proof. We first consider a simple case that, for each E_l, there exists a face $\mathrm{F}_l \subset \partial G$ such that F_l contains E_l and the intersection of F_l with the other edges is connected with E_l (a particular case is that the face F_l does not adjoin the other edges). For each E_l, let E'_l be the intersection of F_l with the other edges. By the assumption, $\mathrm{E}_l \cup \mathrm{E}'_l$ is a union of several connected edges of the face F_l. Regarding $\mathrm{E}_l \cup \mathrm{E}'_l$ as the edge E in the proof of Lemma 4.2 and almost repeating the proof process (but using Lemma 3.1 for $\Gamma = \cup_{l=1}^{N}\mathrm{F}_l$), we can build the desired results.

If the above condition is not met, the proof of this lemma is a bit technical. Without loss of generality, we assume that this condition is not satisfied for each E_l (An example is that $\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3, \mathrm{E}_4$ are just four parallel edges of a cube G, see Figure 4). This means that, for each edge E_l, any face containing E_l must contain another different edge $\mathrm{E}_{l'}$ that is not connected with E_l. In this situation, the above proof is not practical since the functions $\tilde{\phi}_{\mathrm{E}}$ and $\tilde{C}_{\mathrm{E}}$ defined in the proof of Lemma 4.2 may not vanish on $\mathrm{E}_{l'}$.

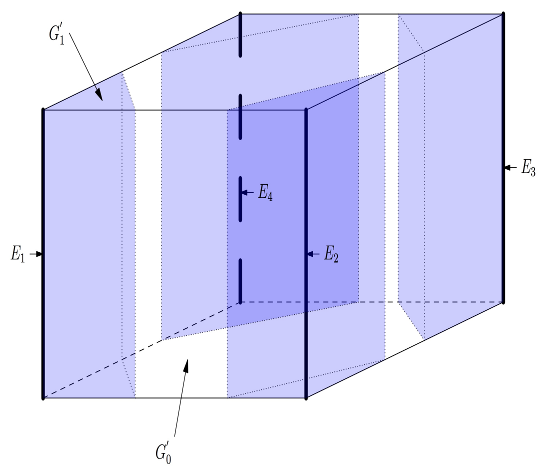

Figure 4: An example with four edges: the left shaded polyhedron denotes the subdomain G_1', the middle polyhedron denotes G_0'.

Notice that the considered edges are disjunct each other, we can decompose G into a union of non-overlapping subdomains $G_0', G_1', \cdots, G_m'$ such that: (i) each subdomain G_l' is a polyhedron with the size $O(1)$; (ii) $\bar{G}_l' \cap \bar{G}_j' = \emptyset$ for $j \neq l$ $(l, j \neq 0)$, and G_0' just has a common face Γ_{0l}' with each G_l' $(l \neq 0)$; (iii) for $l = 1, \cdots, m$, the subdomain G_l' contains E_l as one of its edges, but the subdomain G_0' does not intersect with any E_l. In general we can not require each subdomain G_l' to be a union of some elements. Because of this, for each l we choose a small perturbation G_l of G_l', where G_l is a union of all the elements K satisfying $\mathrm{meas}(K \cap G_l) \geq \frac{1}{2}\mathrm{meas}(K)$ ($\mathrm{meas}(D)$ denotes the measure (i.e., volume) of the domain D). It is clear that G_l is not a usual polyhedron since $\Gamma_{0l} = \bar{G}_0 \cap \bar{G}_l$ is not a plane face yet. Fortunately, all the subdomains $\{G_l\}$ still constitute a union of G and keep the other properties of $\{G_l'\}$.

For each E_l, we use Lemma 4.2 to build a Helmholtz decomposition

$$\mathbf{v}_h = \mathbf{r}_h \mathbf{w}_h^{(l)} + \nabla p_h^{(l)} + \mathbf{R}_h^{(l)} \quad \text{on} \quad G, \tag{4.42}$$

with $\mathbf{w}_h^{(l)}$ and $p_h^{(l)}$ vanishing on E_l (but may not vanishing on the other edges). The decomposition is stable with a logarithmical factor. Let d_0 be a given positive number independent of h. For $l = 1, \cdots, m$, we choose a ball D_l containing G_l such that $dist(\partial D_l, \partial G_l) \geq d_0$ and D_l does not intersect with any G_j for $j \neq 0, l$. Since G_l is a small perturbation of the polyhedron G'_l, there exists an extension $\tilde{\mathbf{w}}^{(l)}$ (resp. $\tilde{p}^{(l)}$) of $\mathbf{w}_h^{(l)}|_{G_l}$ (resp. $p_h^{(l)}|_{G_l}$) such that: (a) $\tilde{\mathbf{w}}^{(l)} \in (H^1(\mathbb{R}^3)^3$ (resp. $\tilde{p}^{(l)} \in H^1(\mathbb{R}^3)$; (b) $\tilde{\mathbf{w}}^{(l)}$ and $\tilde{p}^{(l)}$ vanish on the outside of D_l; (c) $\|\tilde{\mathbf{w}}^{(l)}\|_{1,G_0} \lesssim \|\mathbf{w}_h^{(l)}\|_{1,G_l}$ and $\|\tilde{p}^{(l)}\|_{1,G_0} \lesssim \|p_h^{(l)}\|_{1,G_l}$. For each l, let $\tilde{\mathbf{R}}_h^{(l)} \in V_h(G)$ denote the standard zero extension of $\mathbf{R}_h^{(l)}|_{G_l}$. Define

$$\tilde{\mathbf{v}}_h^{(0)} = \mathbf{v}_h - \sum_{l=1}^{m} (\mathbf{r}_h \tilde{\mathbf{w}}^{(l)} + \nabla \tilde{p}^{(l)} + \tilde{\mathbf{R}}_h^{(l)}) \quad \text{on} \quad G_0. \tag{4.43}$$

It is clear that $\tilde{\mathbf{v}}_h^{(0)} \in H(\mathbf{curl}; G_0)$. Since $\tilde{\mathbf{w}}^{(l)}$, $\tilde{p}^{(l)}$ and $\tilde{\mathbf{R}}_h^{(l)}$ vanish on $\bar{G}_j$ for $j \neq 0, l$, by (4.42) we have $\tilde{\mathbf{v}}_h^{(0)} \times \mathbf{n} = \mathbf{0}$ on Γ_{0l} for $l = 1, \cdots, m$. By Lemma 3.1 (refer to Remark 3.1), $\tilde{\mathbf{v}}_h^{(0)}$ admits a stable decomposition

$$\tilde{\mathbf{v}}_h^{(0)} = \tilde{\mathbf{w}}^{(0)} + \nabla \tilde{p}^{(0)} \quad \text{on} \quad G_0 \tag{4.44}$$

for $\tilde{\mathbf{w}}^{(0)} \in (H^1(G_0))^3$ and $\tilde{p}^{(0)} \in H^1(G_0)$, with $\tilde{\mathbf{w}}^{(0)}$ and $\tilde{p}^{(0)}$ vanishing on Γ_{0l} for $l = 1, \cdots, m$. Combing (4.43) and (4.44), we have

$$\mathbf{v}_h = \tilde{\mathbf{w}}^{(0)} + \nabla \tilde{p}^{(0)} + \sum_{l=1}^{m} (\mathbf{r}_h \tilde{\mathbf{w}}^{(l)} + \nabla \tilde{p}^{(l)} + \tilde{\mathbf{R}}_h^{(l)}) \quad \text{on} \quad G_0.$$

Thus

$$\mathbf{v}_h = \mathbf{r}_h (\tilde{\mathbf{w}}^{(0)} + \sum_{l=1}^{m} \tilde{\mathbf{w}}^{(l)}) + \nabla p_h^{(0)} + \sum_{l=1}^{m} \tilde{\mathbf{R}}_h^{(l)} \quad \text{on} \quad G_0 \tag{4.45}$$

with $p_h^{(0)}$ satisfying $\nabla p_h^{(0)} = \mathbf{r}_h \nabla (\tilde{p}^{(0)} + \sum_{l=1}^{m} \tilde{p}^{(l)})$.

Let $\Pi_h : (H^1(G_0))^3 \to (Z_h(G_0))^3$ denote the Scott-Zhang interpolation operator, which can preserve the values of a linear polynomial on some elements of the boundary ∂G_0. Define

$$\mathbf{w}_h^{(0)} = \Pi_h (\tilde{\mathbf{w}}^{(0)} + \sum_{l=1}^{m} \tilde{\mathbf{w}}^{(l)}) \quad \text{and} \quad \mathbf{R}_h^{(0)} = \mathbf{r}_h (I - \Pi_h)(\tilde{\mathbf{w}}^{(0)} + \sum_{l=1}^{m} \tilde{\mathbf{w}}^{(l)}) + \sum_{l=1}^{m} \tilde{\mathbf{R}}_h^{(l)}.$$

Then (4.45) can be written as

$$\mathbf{v}_h = \mathbf{r}_h \mathbf{w}_h^{(0)} + \nabla p_h^{(0)} + \mathbf{R}_h^{(0)} \quad \text{on} \quad G_0. \tag{4.46}$$

It is clear that $\mathbf{w}_h^{(0)} = \mathbf{w}_h^{(l)}$ and $p_h^{(0)} = p_h^{(l)}$ on Γ_{0l} for $l = 1, \cdots, m$, which implies that $\mathbf{R}_h^{(0)} \times \mathbf{n} = \mathbf{R}_h^{(l)}$ on Γ_{0l} for $l = 1, \cdots, m$. Thus we naturally define $\mathbf{w}_h = \mathbf{w}_h^{(l)}$, $p_h = p_h^{(l)}$ and $\mathbf{R}_h = \mathbf{R}_h^{(l)}$ on G_l for $l = 0, 1, \cdots, m$, and we have $\mathbf{w}_h \in (Z_h(G))^3$, $p_h \in Z_h(G)$ and $\mathbf{R}_h \in V_h(G)$, which have the zero degrees of freedom on all the edges E_l. It is easy to see from (4.42) and (4.46) that the desired Helmholtz decomposition is valid for the defined functions. Besides, we can verify that the resulting Helmholtz decomposition is also stable with a logarithmical factor. ♯

We can replace the edges in Lemma 4.6 by vertices, and we have the following lemma

Lemma 4.7 *Let $\mathbf{v}_h \in V_h(G)$ and $\mathrm{V}_1, \cdots, \mathrm{V}_m$ be different vertices of G. Then there exist functionals $\mathcal{F}_l$ $(l = 1, \cdots, m)$ defined on $V_h(G)$ such that every function $\mathbf{v}_h$ satisfying the m constraints $\mathcal{F}_l \mathbf{v}_h = 0$ $(l = 1, \cdots, m)$ can be decomposed as*

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h$$

for some $p_h \in Z_h(G)$ and $\mathbf{w}_h \in (Z_h(G))^3$ and $\mathbf{R}_h \in V_h(G)$, with p_h and $\mathbf{w}_h$ vanishing at each vertex V_l. Moreover, we have

$$\|\mathbf{w}_h\|_{1,G} \lesssim \log(1/h) \|\mathbf{curl}\, \mathbf{v}_h\|_{0,G}, \quad \|p_h\|_{1,G} \lesssim \log(1/h) \mathbf{v}_h\|_{\mathbf{curl},G} \tag{4.47}$$

and

$$h^{-1} \|\mathbf{R}_h\|_{0,G} \lesssim \log(1/h) \|\mathbf{curl}\, \mathbf{v}_h\|_{0,G}. \tag{4.48}$$

Proof. At first we assume that , for each V_l, there exists a (closed) face $\mathrm{F}_l \subset \partial G$ such that F_l contains V_l but $\mathrm{F}_l \cap \mathrm{F}_i = \emptyset$ for any $i \neq l$. For each V_l, let $\tilde{\phi}_{\mathrm{V}_l}$ and $\tilde{C}_{\mathrm{V}_l}$ be the functions defined as in the proof of Lemma 4.3. Define

$$\hat{\mathbf{v}}_h = \mathbf{v}_h - \sum_{l=1}^{m} (\nabla \tilde{\phi}_{\mathrm{V}_l} + \mathbf{r}_h \tilde{C}_{\mathrm{V}_l}).$$

Then $\hat{\mathbf{v}}_h \cdot \mathbf{t}_{\partial \mathrm{F}_l} = 0$ on $\partial \mathrm{F}_l$ for $l = 1, \cdots, m$. As in Lemma 4.3, we use Lemma 4.1 to build a decomposition of $\hat{\mathbf{v}}_h$ and further get a decomposition of $\mathbf{v}_h$. Let $\mathrm{E}_l \subset \partial \mathrm{F}_l$ be an edge (or a union of several edges) of F_l. For each l, define a functional $\mathcal{F}_l$ as $\mathcal{F}_l \mathbf{v}_h = \gamma_{\mathrm{E}_l}(\phi_{\partial \mathrm{F}_l})$, where $\phi_{\partial \mathrm{F}_l}$ is defined as in the proof of Lemma 4.3 and $\gamma_{\mathrm{E}_l}(\phi_{\partial \mathrm{F}_l})$ denotes the average of the function $\phi_{\partial \mathrm{F}_l}$ on E_l. If $\mathbf{v}_h$ satisfies $\mathcal{F}_l \mathbf{v}_h = 0$ for $l = 1, \cdots, m$, then the functions $\tilde{\phi}_{\mathrm{V}_l}$ and $\tilde{C}_{\mathrm{V}_l}$ vanish at all the vertices $\mathrm{V}_1, \cdots, m$, which implies that the resulting decomposition is the desired decomposition of $\mathbf{v}_h$. Moreover, the norm $\|\tilde{\phi}_{\mathrm{V}_l}\|_{0,G}$ has nearly optimal estimate because of the constraint $\mathcal{F}_l \mathbf{v}_h = 0$ $(l = 1, \cdots, m)$ and so $\|p_h\|_{1,G}$ satisfies the desired estimate.

Now we consider the case that, for some vertex V_l, any face containing V_l at least contains another vertex V_i $(i \neq l)$. Let F_l denote such a face that contains both V_l and V_i. For this face F_l and the vertex V_l, we define a function $\phi_{\partial \mathrm{F}_l}$ as in the proof of Lemma 4.3 and define a functional $\mathcal{F}_l$ in the above way. For the vertex V_i, define a functional $\mathcal{F}_i$ by $\mathcal{F}_i \mathbf{v}_h = \phi_{\partial \mathrm{F}_l}(t_i)$, where t_i is the arc-length of the vertex V_i. Then the extension $\tilde{\phi}_{\mathrm{V}_l}$ of $\phi_{\partial \mathrm{F}_l}$ vanishes at all the vertices $\mathrm{V}_1, \cdots, \mathrm{V}_m$ if $\mathbf{v}_h$ satisfies the constraints $\mathcal{F}_l \mathbf{v}_h = \mathcal{F}_i \mathbf{v}_h = 0$. We can define a function $C_{\partial \mathrm{F}_l}$ as in the proof of Lemma 4.3 such that $C_{\partial \mathrm{F}_l}$ vanishes at both V_l and V_i and satisfies some constraints. We point out that, if the face F_l contains more than two vertices in $\mathrm{V}_1, \cdots, \mathrm{V}_m$, the number of the degrees of freedom of $C_{\partial \mathrm{F}_l}$ may be not enough to satisfy all the constraints in the proof of Lemma 4.3. In this case, we have to choose an auxiliary node $\mathrm{V}_{i'}$ at a suitable position of $\partial \mathrm{F}$ and add three degrees of freedom of $C_{\partial \mathrm{F}_l}$ at $\mathrm{V}_{i'}$. With the zero extension $\tilde{C}_{\mathrm{V}_l}$ of $C_{\partial \mathrm{F}_l}$, we define $\hat{\mathrm{v}}_{h,\mathrm{V}_l} = \mathbf{v}_h - (\nabla \tilde{\phi}_{\mathrm{V}_l} + \mathbf{r}_h \tilde{C}_{\mathrm{V}_l})$. Then we have $\hat{\mathrm{v}}_{h,\mathrm{V}_l} \cdot \mathbf{t}_{\partial \mathrm{F}_l} = 0$ on $\partial \mathrm{F}_l$. Thus we can use Lemma 4.1 to build a decomposition of $\hat{\mathbf{v}}_{h,\mathrm{V}_l}$ and we further get the desired decomposition of $\mathbf{v}_h$ as in the proof of Lemma 4.3. ♯

Remark 4.4 *In the proofs of Lemma 4.1-Lemma 4.7, our main ideas are to transform the problem for vanishing on a vertex V or an edge E into the problem to vanish on a face F of G and then to use the regular Helmholtz decompositions given in Lemma 3.1, which can preserve zero trace on this face.*

Remark 4.5 *In the previous results, a vertex is essential different from an edge. This phenomenon can be intuitively explained as follows: the value of an edge finite element* *function $\mathbf{v}_h$ at a vertex is not uniquely defined, but the two nodal finite element functions defined by the Helmholtz decomposition of $\mathbf{v}_h$ are required to vanish at the vertex. Thus there is a gap between $\mathbf{v}_h$ and the nodal finite element functions, which needs to be filled by a constraint of $\mathbf{v}_h$.*

Notice that, in all the previous Lemmas, a connected union of several edges on one face has no essential difference from an edge. For simplicity of exposition, an "edge" is always understood as an "edge" or a "connected union of edges on one face" in the rest of this paper.

By using Lemma 4.4-Lemma 4.7, we can easily prove the following main result

Theorem 4.1 *Let Γ be a (may be non-connected) union of some vertices, edges and faces of G. Assume that the vector-valued function $\mathbf{v}_h \in V_h(G)$ has zero degree of freedom $\lambda_e(\mathbf{v}_h) = \mathbf{0}$ for all $e \subset \Gamma$. Then, for each vertex $\mathrm{V} \in \Gamma$, there exists a functional $\mathcal{F}_{\mathrm{V}}$ defined on $V_h(G)$ such that, if $\mathbf{v}_h$ satisfies all the constraints $\mathcal{F}_{\mathrm{V}}\mathbf{v}_h = 0$ ($\forall \mathrm{V} \in \Gamma$), the function $\mathbf{v}_h$ admits a decomposition*

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h$$

for some $p_h \in Z_h(G)$, $\mathbf{w}_h \in (Z_h(G))^3$ and $\mathbf{R}_h \in V_h(G)$ such that $p_h = 0$ and $\mathbf{w}_h = \mathbf{0}$ on Γ, and $\lambda_e(\mathbf{R}_h) = \mathbf{0}$ for all $e \subset \Gamma$. Moreover, we have

$$\|\mathbf{w}_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}, \quad \|p_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{v}_h\|_{\mathbf{curl},G} \tag{4.49}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{0,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}. \tag{4.50}$$

In particular, when there is no vertex in Γ, the additional constraints are unnecessary.

$\sharp$

Remark 4.6 *Notice that the second estimate in (4.49) is different from that in (3.14). In fact, if Γ indeed contains an edge or a vertex, then the L^2 stability in (3.14) seems not valid yet.*

5 Regular Helmholtz decompositions on some non-Lipchitz domains

In this section we try to extend some results in the previous section to non-Lipchitz domains. Let G_1 and G_2 be two intersecting convex polyhedra, and set $G = \bar{G}_1 \cup \bar{G}_2$. We consider two particular cases: (1) the intersection $\partial G_1 \cap \partial G_2$ is just the common edge of G_1 and G_2; (2) the intersection $\partial G_1 \cap \partial G_2$ is just the common vertex of G_1 and G_2. For the two cases, G is not a Lipchitz domain.

The following two theorems can be viewed as extensions of Lemma 3.1 to the case of non-Lipchitz domains.

Theorem 5.1 *Let G be defined above, with $\bar{G}_1 \cap \bar{G}_2$ being the common edge of G_1 and G_2, and let Γ be a union of some faces of G_1 and G_2. Assume that the vector-valued function $\mathbf{v}_h \in V_h(G)$ satisfies $\mathbf{v}_h \times \mathbf{n} = \mathbf{0}$ on Γ, then $\mathbf{v}_h$ admits a decomposition*

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h, \tag{5.1}$$

for some $p_h \in Z_h(G)$, $\mathbf{w}_h \in (Z_h(G))^3$ and $\mathbf{R}_h \in V_h(G)$ such that $p_h = 0$ and $\mathbf{w}_h = \mathbf{0}$ on Γ, and $\mathbf{R}_h \times \mathbf{n} = \mathbf{0}$ on Γ. Moreover, we have

$$\|\mathbf{w}_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}, \quad \|p_h\|_{1,G} \lesssim \log(1/h)\|\mathbf{v}_h\|_{\mathbf{curl},G} \tag{5.2}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{0,G} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,G}. \tag{5.3}$$

In particular, when $\bar{G}_1 \cap \bar{G}_2 \subset \Gamma \cap \partial G_i$ for $i = 1, 2$, the logarithm factor in the above estimates can be dropped.

Proof. For convenience, set $\mathrm{E} = \bar{G}_1 \cap \bar{G}_2$ and $\Gamma_i = \Gamma \cap \partial G_i$ $(i = 1, 2)$. We prove the theorem according to three different position relations between E and Γ_i $(i = 1, 2)$.

(i) $\mathrm{E} \subset \Gamma_i$ for $i = 1, 2$. We use Lemma 3.1 to build a decomposition of $\mathbf{v}_h|_{G_i}$ independently for $i = 1, 2$. Then the resulting functions $p_{h,i}$ and $\mathbf{w}_{h,i}$ vanish on Γ_i and so they also vanish on E. Thus we can directly extend $p_{h,i}$ and $\mathbf{w}_{h,i}$ into another domain by zero to get the global extension of $\mathbf{v}_h$ on G. In this case, there is no logarithm factor in the stability estimates.

(ii) E is contained in only one of Γ_1 and Γ_2, for example, Γ_1. We use Lemma 3.1 to build a decomposition of $\mathbf{v}_h|_{G_1}$, but use Theorem 4.1 to get a decomposition of $\mathbf{v}_h|_{G_2}$ with $\Gamma = \Gamma_2 \cup \mathrm{E}$ (notice that $\mathbf{v}_h|_{G_2}$ vanishes on Γ_2 and E since $\mathrm{E} \subset \Gamma_1 \subset \Gamma$). Then the desired decomposition can be built as in the above situation.

(iii) $\mathrm{E} \cap \Gamma_i = \emptyset$ for $i = 1, 2$. We first use Lemma 3.1 to build the decomposition of $\mathbf{v}_h|_{G_1}$

$$\mathbf{v}_h = p_{h,1} + \mathbf{r}_h \mathbf{w}_{h,1} + \mathbf{R}_{h,1} \quad \text{on} \ \ G_1, \tag{5.4}$$

with $p_{h,1} \in Z_h(G_1)$ and $\mathbf{w}_{h,1} \in (Z_h(G_1))^3$ vanishing on Γ_1 (and so $\mathbf{R}_{h,1} \times \mathbf{n} = \mathbf{0}$ on Γ_1). Moreover, we have

$$\|\mathbf{w}_{h,1}\|_{1,G_1} \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,G_1}, \quad \|p_{h,1}\|_{1,G_1} \lesssim \|\mathbf{v}_h\|_{\mathbf{curl},G_1} \tag{5.5}$$

and

$$h^{-1}\|\mathbf{R}_{h,1}\|_{0,G_1} \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,G_1}. \tag{5.6}$$

Notice that $p_{h,1}$, $\mathbf{w}_{h,1}$ and $\mathbf{R}_{h,1}$ may be not vanish on E since $\mathrm{E} \cap \Gamma_1 = \emptyset$. We extend $p_{h,1}$, $\mathbf{w}_{h,1}$ and $\mathbf{R}_{h,1}$ into G_2 such that the resulting extensions $\tilde{p}_{h,1} \in Z_h(G)$, $\tilde{\mathbf{w}}_{h,1} \in (Z_h(G))^3$ and $\tilde{\mathbf{R}}_{h,1} \in V_h(G)$ have zero degrees of freedom on the nodes or fine edges in $G_2 \backslash \mathrm{E}$. Define

$$\mathbf{v}_h^* = \mathbf{v}_h - (\tilde{p}_{h,1} + \mathbf{r}_h \tilde{\mathbf{w}}_{h,1} + \tilde{\mathbf{R}}_{h,1}) \quad \text{on} \ \ G. \tag{5.7}$$

Then $\mathbf{v}_h^*|_{G_2}$ vanishes on Γ_2 and E, and it admits the decomposition by Lemma 4.4

$$\mathbf{v}_h^* = p_{h,2}^* + \mathbf{r}_h \mathbf{w}_{h,2}^* + \mathbf{R}_{h,2}^* \quad \text{on} \ \ G_2, \tag{5.8}$$

with $p_{h,2}^* \in Z_h(G_2)$ and $\mathbf{w}_{h,2}^* \in (Z_h(G_2))^3$ vanishing on Γ_2 and E. Moreover, we have

$$\|\mathbf{w}_{h,2}^*\|_{1,G_2} \lesssim \|\mathbf{curl}\,\mathbf{v}_h^*\|_{0,G_2}, \quad \|p_{h,2}^*\|_{1,G_2} \lesssim \|\mathbf{v}_h^*\|_{\mathbf{curl},G_2} \tag{5.9}$$

and

$$h^{-1}\|\mathbf{R}_{h,2}^*\|_{0,G_2} \lesssim \|\mathbf{curl}\,\mathbf{v}_h^*\|_{0,G_2}. \tag{5.10}$$

We extend $p_{h,2}^*$, $\mathbf{w}_{h,2}^*$ and $\mathbf{R}_{h,2}^*$ into G_1 by zero. Since these functions vanish on E, the resulting extensions $\tilde{p}_{h,2}^*$, $\tilde{\mathbf{w}}_{h,2}^*$ and $\tilde{\mathbf{R}}_{h,2}^*$ satisfy $\tilde{p}_{h,2}^* \in Z_h(G)$, $\tilde{\mathbf{w}}_{h,2}^* \in (Z_h(G))^3$ and $\tilde{\mathbf{R}}_{h,2}^* \in V_h(G)$. Define

$$p_h = \tilde{p}_{h,1} + \tilde{p}_{h,2}^*, \ \ \mathbf{w}_h = \tilde{\mathbf{w}}_{h,1} + \tilde{\mathbf{w}}_{h,2}^* \ \ \text{and} \ \ \mathbf{R}_h = \tilde{\mathbf{R}}_{h,1} + \tilde{\mathbf{R}}_{h,2}^*.$$

Then the decomposition (5.1) follows by (5.7) and (5.8).

In an analogous way with Step 2 in the proof of Theorem 3.1, we can verify the estimates (5.2) and (5.3) by using (5.5)-(5.6) and (5.9)-(5.10), combining the "edge" lemma in [31]. ♯

When $\bar{G}_1 \cap \bar{G}_2$ is just the common vertex V of G_1 and G_2, it becomes more complicated to study Helmholtz decomposition of $\mathbf{v}_h$ on $G = \bar{G}_1 \cup \bar{G}_2$. Let Γ be a union of some faces of G_1 and G_2, and $\mathbf{v}_h$ has zero tangential trace on Γ. We first consider a simple case: one of the two sets $\Gamma_i = \Gamma \cap \partial G_i$ $(i = 1, 2)$, for example Γ_1, is just the empty set, i.e., Γ is a union of some faces that are contained in ∂G_2 and do not contain V. For this case, the results can be built as in the case (iii) of the proof of Theorem 5.1: we first use Lemma 3.1 to define a decomposition of $\mathbf{v}_h|_{G_2}$, then we extend the resulting functions into G_1, and we further use Lemma 4.3 to define a decomposition of the remainder of $\mathbf{v}_h$ on G_1.

In the following, we consider the case with $\Gamma_i \neq \emptyset$ $(i = 1, 2)$ and $\mathrm{V} \notin \Gamma$. For this case, similar results with Theorem 5.1 can not be obtained except that some additional condition on $\mathbf{v}_h$ is met. Let $\mathrm{F}_1 \subset \partial G_1$ and $\mathrm{F}_2 \subset \partial G_2$ denote two faces containing V. As in the proof of Lemma 4.3, we define

$$C_{\partial \mathrm{F}_i} = \frac{1}{l_i} \int_0^{l_i} (\mathbf{v}_h \cdot \mathbf{t}_{\partial \mathrm{F}_i})(s)\, ds, \quad \phi_{\partial \mathrm{F}_i}(t) = \int_0^t (\mathbf{v}_h \cdot \mathbf{t}_{\partial \mathrm{F}_i} - C_{\partial \mathrm{F}_i})(s) ds + c_i\,, \quad \forall\, t \in [0, l_i]\,,$$

where l_i is the length of $\partial \mathrm{F}_i$ and $t = 0$ (and $t = l_i$) corresponds the vertex V. For $i = 1, 2$, the constant c_i are chosen such that $\gamma_{\mathrm{E}_i}(\phi_{\partial \mathrm{F}_i}) = 0$, where $\mathrm{E}_i \subset \partial \mathrm{F}_i$ is an edge or a union of some edges.

Theorem 5.2 *Let G be a union of G_1 and G_2, with $\bar{G}_1 \cap \bar{G}_2$ being the common vertex V of G_1 and G_2, and let Γ be a union of some faces of G_1 and G_2 such that $\mathrm{V} \notin \Gamma$. Assume that the vector-valued function $\mathbf{v}_h \in V_h(G)$ satisfies $\mathbf{v}_h \times \mathbf{n} = \mathbf{0}$ on Γ. If the following additional condition is met: there are two faces F_i $(\subset \partial G_i)$ $(i = 1, 2)$ containing V such that $\phi_{\partial \mathrm{F}_1}(t_{\mathrm{V}}^{(1)}) = \phi_{\partial \mathrm{F}_2}(t_{\mathrm{V}}^{(2)})$, where the function $\phi_{\partial \mathrm{F}_i}$ is defined above and the number $t_{\mathrm{V}}^{(i)}$ is the arc-length coordinate corresponding to the point $\mathrm{V} \in \partial \mathrm{F}_i$, then $\mathbf{v}_h$ has a decomposition*

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h \tag{5.11}$$

for some $p_h \in Z_h(G)$, $\mathbf{w}_h \in (Z_h(G))^3$ and $\mathbf{R}_h \in V_h(G)$ such that $p_h = 0$, $\mathbf{w}_h = \mathbf{0}$ and $\mathbf{R}_h \times \mathbf{n} = \mathbf{0}$ on Γ. Moreover, we have

$$\|\mathbf{w}_h\|_{1,D} \lesssim \log(1/h) \|\mathbf{curl}\, \mathbf{v}_h\|_{0,G}, \quad \|p_h\|_{1,G} \lesssim \log(1/h) \|\mathbf{v}_h\|_{0,G} \tag{5.12}$$

and

$$h^{-1} \|\mathbf{R}_h\|_{0,G} \lesssim \log(1/h) \|\mathbf{curl}\, \mathbf{v}_h\|_{0,G}. \tag{5.13}$$

Proof. Without loss of generality, we only consider the case with $\mathrm{F}_i \cap \Gamma = \emptyset$ $(i = 1, 2)$. As in the proof of Lemma 4.3, we define the extensions $\tilde{\phi}_{\mathrm{V}}^{(i)}$ and $\tilde{C}_{\mathrm{V}}^{(i)}$ of $\phi_{\partial \mathrm{F}_i}$ and $C_{\partial \mathrm{F}_i}$ respectively, but define the values of $\tilde{\phi}_{\mathrm{V}}^{(i)}$ as zero at all the nodes in G_i except on $\partial \mathrm{F}_i$. Set

$$\hat{\mathbf{v}}_{h,\mathrm{V}}^{(i)} = \mathbf{v}_h|_{G_i} - (\tilde{\phi}_{\mathrm{V}}^{(i)} + \mathbf{r}_h \tilde{C}_{\mathrm{V}}^{(i)}) \quad \text{on} \quad G_i.$$

Then we have $\lambda_e(\hat{\mathbf{v}}_{h,\mathrm{V}}^{(i)}) = 0$ for any $e \subset \partial \mathrm{F}_i \cup (\Gamma \cap \partial G_i)$ by the definitions of $\tilde{\phi}_{\mathrm{V}}^{(i)}$ and $\tilde{C}_{\mathrm{V}}^{(i)}$. Thus we can use **Corollary 4.1** for $\hat{\mathbf{v}}_{h,\mathrm{V}}^{(i)}$ to build a decomposition of $\mathbf{v}_h|_{G_i}$ and the corresponding estimates by the condition $\gamma_{\mathrm{E}_i}(\phi_{\partial \mathrm{F}_i}) = 0$ (refer to the proof of Lemma 4.3). Using the definitions of the functions $p_{h,i}$ and $\mathbf{w}_{h,i}$ in the decomposition of $\mathbf{v}_h|_{G_i}$, together with the condition $\phi_{\partial \mathrm{F}_1}(t_{\mathrm{V}}^{(1)}) = \phi_{\partial \mathrm{F}_2}(t_{\mathrm{V}}^{(2)})$, yields $p_{h,1} = p_{h,2}$ and $\mathbf{w}_{h,i} = \mathbf{0}$ at the vertex V. Therefore, we can naturally define a decomposition of $\mathbf{v}_h$ on the global G and obtain the desired stability estimates. ♯

Remark 5.1 *The condition $\phi_{\partial F_1}(t_V^{(1)}) = \phi_{\partial F_2}(t_V^{(2)})$ in the above theorem seems absolutely necessary (a counterexample can be constructed as in Remark 4.3). But, when $V \subset \Gamma \cap \partial G_i$ for $i = 1, 2$, the additional condition in Theorem 5.2 is unnecessary and the logarithm factor in (5.12)-(5.13) can be dropped. In fact, we can use Lemma 3.1 to build a decomposition of $\mathbf{v}_h|_{G_i}$ independently for $i = 1, 2$, such that the resulting functions $p_{h,i}$ and $\mathbf{w}_{h,i}$ vanish on $\Gamma \cap \partial G_i$ that contains V. Thus we can directly extend $p_{h,i}$ and $\mathbf{w}_{h,i}$ into another domain by zero to get the global extension of $\mathbf{v}_h$ on G and obtain the desired stability estimates.*

We can also consider the case that G is a union of more polyhedrons. Let $G_1, G_2, \cdots, G_s$ be Lipchitz polyhedrons that may be non-convex, and let G be a union of $G_1, G_2, \cdots, G_s$ (then G is a non-Lipchitz domain) such that the intersection of any two polyhedrons in $G_1, G_2, \cdots, G_s$ is just the same vertex of them, i.e., $\bar{G}_i \cap \bar{G}_j = V$ (a vertex) for $i \neq j$ (of course, $\bar{G}_1 \cap \bar{G}_2 \cap \cdots \cap \bar{G}_s = V$).

Theorem 5.3 *Let G_i ($i = 1, \cdots, s$; $s \geq 3$) and G be defined above, and let Γ denote a union of some faces of $G_1, \cdots, G_s$. Assume that the vector-valued function $\mathbf{v}_h \in V_h(G)$ has the zero tangential trace $\mathbf{v}_h \times \mathbf{n} = \mathbf{0}$ on Γ. Then there exist $s-1$ functionals $\mathcal{F}_i$ such that, if $\mathbf{v}_h$ satisfies the constraints $\mathcal{F}_i \mathbf{v}_h = 0$ for $i = 1, \cdots, s-1$, the function $\mathbf{v}_h$ admits a Helmholtz decomposition like (5.11) and the resulting functions satisfy the conditions and estimates in Theorem 5.2.*

Proof. Let F_i be a face satisfying $V \in F_i \subset \partial G_i$ and define a function $\phi_{\partial F_i}$ as in Theorem 5.2 ($i = 1, \cdots, s$). We define the functional $\mathcal{F}_i$ by

$$\mathcal{F}_i \mathbf{v}_h = \phi_{\partial F_{i+1}}(V) - \phi_{\partial F_i}(V) \quad (i = 1, \cdots, s-1);$$

Then we can prove the desired results in an analogous way with the proof of Theorem 5.2. ♯

Remark 5.2 *The theorems given in this section still hold if there are some edges in Γ.*

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