A second order finite difference-spectral method for space fractional diffusion equation

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Abstract

A high order finite difference-spectral method is derived for solving space fractional diffusion equations, by combining the second order finite difference method in time and the spectral Galerkin method in space. The stability and error estimates of the temporal semidiscrete scheme are rigorously discussed. Then the convergence order of this proposed method is proved to be $O(\tau^2 + N^{\alpha-m})$ in L^2-norm, where τ, N, α and m are the time step size, polynomial degree, fractional derivative index and regularity of the exact solution respectively. Numerical experiments are carried out to demonstrate the theoretical analysis.

Keywords: Space fractional diffusion equation; Crank-Nicolson scheme; Spectral method; Stability; Convergence.

1 Introduction

Fractional calculus has gained considerable popularity and importance due to its attractive applications as a new modeling tool in an variety of scientific and engineering fields, such as viscoelasticity [9], hydrology [2, 3], finance [8, 12, 14], and system control [13]. These fractional models, described in the form of fractional differential equations, tend to be much more appropriate for the description of memory and hereditary properties of various materials and processes than the traditional integer-order models [16]. In particular, modeling of anomalous diffusion in a specific type of porous medium is one of the most significant applications of fractional derivatives [2, 3].

In the last decade, a number of numerical methods have been developed to solve the space fractional diffusion equation, the time fractional diffusion equation, and the time-space fractional one (see, e.g., [7, 17, 21, 22, 23, 25]). A most classical one for the space fractional diffusion equation is the second order fractional Crank-Nicolson method proposed by Tadjeran et al. in [19, 20], where the Richardson extrapolation technique is used to the first order shift Grünwald formula for space fractional derivative. An interesting $2-\gamma$ ($0<\gamma<1$) order scheme in temporal direction for the time fractional diffusion equation was constructed by Lin and Xu [11], and Sun and Wu [18] respectively. But when $\gamma \to 1$, this scheme would become backward Euler method, and also the order would reduce to only 1.

In addition, to obtain the variational formulation of fractional differential equations, Ervin and Roop [5] originally introduced some fractional derivative spaces and their norms. Based on these fractional spaces, finite element method [4, 6, 24] and spectral method [10] are designed for fractional diffusion equations. In [6], Ervin et al. provided a forward Euler difference-finite element method

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to solve the one-dimension space fractional diffusion equation, and thus the accuracy in temporal direction had first order convergence. A time-space spectral method for the time fractional diffusion equation was discussed by Li and Xu in [10], this method had spectral accuracy both in time and space.

This paper devotes to design a new high order numerical method for the following space fractional diffusion equation

$$\frac{\partial u(x,t)}{\partial t} - a{}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\mathbf{2}\alpha}_{\mathbf{x}}u(x,t) = f(x,t),\ x \in \Omega, t \in I, \tag{1}$$

equipped with initial-boundary conditions $u(x,0) = \phi(x)$, $u(-1,t) = u(1,t) = 0$, where $\Omega = (-1,1)$, $I = (0,T]$, diffusion coefficient $a > 0$, fractional derivative index $0.5 < \alpha < 1$. ${}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\alpha}_{\mathbf{x}}u(x,t)$ is the space left Riemann-Liouville fractional derivative of $u(x,t)$ defined as

$${}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\alpha}_{\mathbf{x}}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-1}^{x}(x-s)^{-\alpha}u(s,t)ds,$$

where $\Gamma(\cdot)$ is the Gamma function.

Specifically, we will use central finite difference to discrete the time partial derivative and exploit the spectral Galerkin method to approximate the space fractional derivative. Based on these discretization methods, a high order numerical method with second order convergence in time and spectral accuracy in space will be obtained.

Since our new numerical method based on variational formulation, the right Riemann-Liouville fractional derivative ${}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\alpha}_{\mathbf{1}}u(x,t)$, defined as follows, will be used.

$${}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\alpha}_{\mathbf{1}}u(x,t) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x}^{1}(s-x)^{-\alpha}u(s,t)ds.$$

In fact, the proposed method can also be applied to solve following two-dimensional space fractional diffusion equation

$$\frac{\partial u(x,y,t)}{\partial t} - a{}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\mathbf{2}\alpha}_{\mathbf{x}}u(x,y,t) - b{}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\mathbf{2}\beta}_{\mathbf{y}}u(x,y,t) = f(x,y,t),$$

with $b > 0$ and $0.5 < \beta < 1$ or even three-dimensional fractional ones. Due to the similarity in theoretical analysis, we will focus on the numerical analysis of the one-dimensional case (1).

The rest of this paper is organized as follows. In Section 2 we introduce some fractional derivative spaces and some basic properties of fractional derivative. The temporal discretization of (1) is discussed in Section 3. Section 4 is devoted to analyze the stability and convergence of the semidiscrete form. In Section 5, we derive the full discretization of space fractional diffusion equation, and the error estimates with two different norms are obtained. Numerical experiments are carried out in Section 6, which verify the effectiveness of our method and support the theoretical analysis. Final section is the concluding remarks.

2 Preparation

In this section, we introduce four kinds of fractional derivative spaces and their corresponding norms; for more detailed discussions, one can refer to [5, 10]. And some basic properties of fractional derivative, which will be used in following sections, are given. For simplicity, we use the expression $b_1 \lesssim b_2$ to mean that $b_1 \leq cb_2$, where c is a positive constant, which is independent of all discretization parameters, but possibly with different values at different places.

Definition 2.1 *Let $\mu > 0$. Define the seminorm*

$$|f(x)|_{J_L^\mu(\Omega)} = \|{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mu} f(x)\|_{L^2(\Omega)}$$

and the norm

$$\|f(x)\|_{J_L^\mu(\Omega)} = \left(\|f(x)\|_{L^2(\Omega)}^2 + |f(x)|_{J_L^\mu(\Omega)}^2\right)^{\frac{1}{2}},$$

and denote $J_L^\mu(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{J_L^\mu(\Omega)}$, where $C_0^\infty(\Omega)$ is the space of smooth functions with compact support in Ω.

Definition 2.2 *Let $\mu > 0$. Define the seminorm*

$$|f(x)|_{J_R^\mu(\Omega)} = \|{}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\mu} f(x)\|_{L^2(\Omega)}$$

and the norm

$$\|f(x)\|_{J_R^\mu(\Omega)} = \left(\|f(x)\|_{L^2(\Omega)}^2 + |f(x)|_{J_R^\mu(\Omega)}^2\right)^{\frac{1}{2}},$$

and let $J_R^\mu(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{J_R^\mu(\Omega)}$.

Definition 2.3 *Let $\mu > 0$, $\mu \neq n - \frac{1}{2}$, $n \in \mathbb{N}$. We define the seminorm*

$$|f(x)|_{J_S^\mu(\Omega)} = \left| \left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mu} f(x), {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\mu} f(x)\right) \right|^{\frac{1}{2}}$$

and the norm

$$\|f(x)\|_{J_S^\mu(\Omega)} = \left(\|f(x)\|_{L^2(\Omega)}^2 + |f(x)|_{J_S^\mu(\Omega)}^2\right)^{\frac{1}{2}}.$$

Then we define $J_S^\mu(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to $\|\cdot\|_{J_S^\mu(\Omega)}$.

Definition 2.4 *Let $\mu > 0$. Define the seminorm*

$$|f(x)|_{H^\mu(\Re)} = \||\omega|^\mu \widehat{f}\|_{L^2(\Re)}$$

and the norm

$$\|f(x)\|_{H^\mu(\Re)} = \left(\|f\|_{L^2(\Re)}^2 + |f|_{H^\mu(\Re)}^2\right)^{\frac{1}{2}},$$

where $\widehat{f}$ is the Fourier transformation of function $f(x)$. And Let $H^\mu(\Re)$ be the closure of space $C_0^\infty(\Re)$ with respect to norm $\|\cdot\|_{H^\mu(\Re)}$.

For the bounded interval Ω, we can introduce following space

$$H^\mu(\Omega) = \{f \in L^2(\Omega) | \exists \tilde{f} \in H^\mu(\Re), \tilde{f}|_\Omega = f\}$$

and norm

$$\|f\|_{H^\mu(\Omega)} = \inf_{\tilde{f} \in H^\mu(\Re), \tilde{f}|_\Omega = f} \|f\|_{H^\mu(\Re)},$$

and $H_0^\mu(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ with respect to norm $\|\cdot\|_{H^\mu(\Omega)}$.

For any $f \in C_0^\infty(\Omega)$, let $\widetilde{f}$ be the extension of f by zero outside Ω. Then we have

Property 2.1 *For $\mu > 0$, it holds that*

$$\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mu} f(x), {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\mu} f(x)\right) = \left({}_{-\infty}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mu} \widetilde{f}(x), {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\infty}^{\mu} \widetilde{f}(x)\right) = cos(\pi\mu)\|{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mu} f(x)\|_{L^2(\Omega)}^2.$$

Maybe the spaces and the norms defined above seem tedious, but interestingly the equivalences of these spaces can be established.

Lemma 2.1 *Let $\mu > 0$, $\mu \neq n - \frac{1}{2}$, $n \in \mathbb{N}$. These spaces $J_L^\mu(\Omega)$, $J_R^\mu(\Omega)$, $J_S^\mu(\Omega)$, and $H_0^\mu(\Omega)$ are equal in the sense that their seminorms as well as norms are equivalent.*

Property 2.2 *For $0 < s < \mu$, we know*

$$|f(x)|_{J_L^s(\Omega)} \lesssim |f(x)|_{J_L^\mu(\Omega)} \text{ and } |f(x)|_{J_R^s(\Omega)} \lesssim |f(x)|_{J_R^\mu(\Omega)}.$$

For the proofs of all above results, we can see [5, 10]. As is well known that the unconditional semigroup property of fractional Riemann-Liouville operator does not hold. However, we can obtain the following additivity for fractional Riemann-Liouville operator under a weak assumption on function $f(x)$.

Lemma 2.2 *For $0 < \beta, \gamma < 1$, if $f(x) \in H^1(\Omega)$, then*

$${}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\beta} \cdot {}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\gamma} f(x) = {}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\beta+\gamma} f(x).$$

Proof. According to the definition of fractional derivative,

$${}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\beta} \cdot {}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\gamma} f(x) = \frac{d}{\Gamma(1-\beta)dx} \int_{-1}^{x} (x-s)^{-\beta} \frac{d}{\Gamma(1-\gamma)ds} \int_{-1}^{s} (s-\tau)^{-\gamma} f(\tau) d\tau ds.$$

We interchange the order of integration, obtaining

$$\begin{aligned}
& {}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\beta} \cdot {}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\gamma} f(x) \\
= & \frac{d}{\Gamma(1-\beta)\Gamma(1-\gamma)dx} \int_{-1}^{x} (x-s)^{-\beta} \left[(s+1)^{-\gamma} f(-1) + \int_{-1}^{s} (s-\tau)^{-\gamma} f'(\tau) d\tau \right] ds \\
= & \frac{f(-1) x^{-\beta-\gamma}}{\Gamma(1-\beta-\gamma)} + \frac{d}{\Gamma(2-\beta-\gamma)dx} \int_{-1}^{x} (x-\tau)^{1-\beta-\gamma} f'(\tau) d\tau \\
= & \begin{cases} \frac{d}{\Gamma(1-\beta-\gamma)dx} \int_{-1}^{x} (x-\tau)^{-\beta-\gamma} f(\tau) d\tau, \ 0 < \beta+\gamma < 1 \\ \frac{d^2}{\Gamma(2-\beta-\gamma)dx^2} \int_{-1}^{x} (x-\tau)^{1-\beta-\gamma} f(\tau) d\tau, \ 1 \leq \beta+\gamma < 2 \end{cases} \\
= & {}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\beta+\gamma} f(x). \quad \square
\end{aligned}$$

Following lemma is very important and will be used in the process of deducting variational formulation of space fractional diffusion equation.

Lemma 2.3 *For $0 < \mu < 1$, then from [16], we have*

$$\int_{-1}^{1} f(x) \left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mu} g(x) \right) dx = \int_{-1}^{1} g(x) \left({}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\mu} f(x) \right) dx,$$

where $f(x)$, $g(x)$, ${}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mu} g(x)$ and ${}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\mu} f(x)$ are all belong to $L^2(\Omega)$.

3 Temporal discretization

Rewrite the space fractional diffusion equation Eq.(1) with its initial-boundary conditions

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} - a\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} u(x,t) = f(x,t),\ x\in\Omega,\ t\in I, \\ u(x,0) = \phi(x),\ x\in\Omega \\ u(-1,t) = u(1,t) = 0,\ t\in I. \end{cases} \tag{2}$$

For the finite difference of temporal direction, let $t_i = i\tau$, $i = 0, 1, \cdots, M$, $\tau = \frac{T}{M}$ is the time step size, and denote $t_{i+\frac{1}{2}} = \dfrac{t_{i+1} + t_i}{2}$. Now we consider Eq.(2) on line $(x, t_{i+\frac{1}{2}})$, namely

$$\frac{\partial u(x,t_{i+\frac{1}{2}})}{\partial t} - a\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} u(x,t_{i+\frac{1}{2}}) = f(x,t_{i+\frac{1}{2}}). \tag{3}$$

Using following difference approximations,

$$\begin{aligned} \frac{\partial u(x,t_{i+\frac{1}{2}})}{\partial t} &= \frac{u(x,t_{i+1}) - u(x,t_n)}{\tau} - R_1^i(x)\tau^2, \\ u(x,t_{i+\frac{1}{2}}) &= \frac{u(x,t_{i+1}) + u(x,t_i)}{2} - R_2^i(x)\tau^2, \end{aligned}$$

where $R_1^i(x)\tau^2$ and $R_2^i(x)\tau^2$ are the truncation errors, we discretize Eq.(3), namely

$$\frac{u(x,t_{i+1}) - u(x,t_i)}{\tau} - a\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} \frac{u(x,t_{i+1}) + u(x,t_i)}{2} = f^{i+\frac{1}{2}} + R^i(x)\tau^2, \tag{4}$$

where $f^{i+\frac{1}{2}} = f(x,t_{i+\frac{1}{2}})$, $R^i(x) = R_1^i(x) - a\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} R_2^i(x)$. Obviously, assume that $u(x,t)$ is smooth enough, then there exists a constant c such that $|R^i(x)| \le c$.

In fact, this discretization idea is similar to the treatment of classical Crank-Nicolson method in temporal direction.

Now rearranging Eq.(4) as follows

$$\begin{aligned} u(x,t_{i+1}) - \frac{a\tau}{2}\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} u(x,t_{i+1}) \quad = \quad & u(x,t_i) + \frac{a\tau}{2}\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} u(x,t_i) \\ & + \tau f^{i+\frac{1}{2}} + R^i(x)\tau^3. \end{aligned} \tag{5}$$

Omitting the truncation term $R^i(x)\tau^3$ in Eq.(5), we obtain the semidiscrete scheme of Eq.(2)

$$u^{i+1} - \frac{a\tau}{2}\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} u^{i+1} = u^i + \frac{a\tau}{2}\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} u^i + \tau f^{i+\frac{1}{2}}, \tag{6}$$

where u^{i+1} represents the approximation of $u(x, t_{i+1})$.

For the first step, i.e., $i = 0$, the semidiscrete problem becomes

$$u^1 - \frac{a\tau}{2}\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} u^1 = \phi + \frac{a\tau}{2}\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} \phi + \tau f^{\frac{1}{2}}, \tag{7}$$

with the boundary value condition $u^1(-1) = u^1(1) = 0$.

When $i \ge 1$, the semidiscrete problem is

$$u^{i+1} - \frac{a\tau}{2}\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} u^{i+1} = u^i + \frac{a\tau}{2}\,{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{2\alpha} u^i + \tau f^{i+\frac{1}{2}}, \tag{8}$$

with the boundary value condition $u^{i+1}(-1) = u^{i+1}(1) = 0$.

Thus Eq.(7) and Eq.(8) combined with their boundary conditions form a complete set of the semidiscrete problem.

Multiplying the Eq.(6) by a function $v \in H_0^\alpha(\Omega)$, integrating over the interval Ω, and by Lemmas 2.2 and 2.3, we obtain the variational formulation as follows, i.e., find a $u^{i+1} \in H_0^\alpha(\Omega)$ such that

$$(u^{i+1}, v) - \frac{a\tau}{2}\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} u^{i+1}, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} v\right) = (u^i, v) + \frac{a\tau}{2}\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} u^{i}, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} v\right) + \tau(f^{i+\frac{1}{2}}, v),\ \forall v \in H_0^\alpha(\Omega). \quad (9)$$

Next we will prove the existence and unique of solution for Problem (9). We first define a new norm for $0.5 < \mu < 1$

$$\|f(x)\|_{J_C^\mu(\Omega)} = \left(\|f(x)\|^2_{L^2(\Omega)} - \frac{a\tau}{2}\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mu} f(x), {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\mu} f(x)\right)\right)^{\frac{1}{2}}.$$

Obviously, for a fixed time step size τ, $\|\cdot\|_{J_C^\mu(\Omega)}$ is equivalent to $\|\cdot\|_{J_S^\mu(\Omega)}$ and $\|\cdot\|_{H^\mu(\Omega)}$.

We assume that u^j $(j = 0, 1, \cdots, i)$ are known and u^{i+1} is unknown, and let

$$\begin{aligned} B(u^{i+1}, v) &= (u^{i+1}, v) - \frac{a\tau}{2}\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} u^{i+1}, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} v\right), \\ g &= u^i + \frac{a\tau}{2}{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha} u^i + \tau f^{i+\frac{1}{2}}, \end{aligned} \quad (10)$$

and

$$F(v) = (g, v).$$

Then the simplified form of Problem (9) reads

$$B(u^{i+1}, v) = F(v),\ \forall v \in H_0^\alpha(\Omega). \quad (11)$$

It can be easily checked that F is continuous over $H_0^\alpha(\Omega)$. In order to guarantee Problem (11) admits a unique solution, by means of the Lax-Milgram theorem we only need to prove the coercivity and continuity of bilinear form $B(\cdot,\cdot)$. According to the non-negativity of norm $\|\cdot\|_{J_C^\alpha(\Omega)}$, we know

$$B(v, v) = (v, v) - \frac{a\tau}{2}\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} v, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} v\right) = \|v\|^2_{J_C^\alpha(\Omega)} \geq 0,$$

i.e., $B(\cdot,\cdot)$ is coercive over $H_0^\alpha(\Omega)$.

Lemma 3.1 *The bilinear form $B(\cdot,\cdot)$ is continuous over $H_0^\alpha(\Omega) \times H_0^\alpha(\Omega)$, namely*

$$|B(u, v)| \lesssim \|u\|_{J_C^\alpha(\Omega)}\|v\|_{J_C^\alpha(\Omega)}.$$

Proof.

$$\begin{aligned} |B(u, v)| &= |(u, v) - \frac{a\tau}{2}\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} u, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} v\right)| = |(u, v)| + \frac{a\tau}{2}|\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} u, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} v\right)| \\ &\lesssim \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \tau|u|_{J_L^\alpha(\Omega)}|v|_{J_R^\alpha(\Omega)} \\ &\lesssim \|u\|_{J_C^\alpha(\Omega)}\|v\|_{J_C^\alpha(\Omega)}. \quad \Box \end{aligned}$$

4 Stability and convergence of the semidiscrete form

Before carrying out the related analysis, we establish following lemma.

Lemma 4.1 *For $0.5 < \alpha < 1$ and τ sufficiently small, if ${}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha} f(x) \in L^2(\Omega)$ and $f(x) \in H_0^\alpha(\Omega)$, then*

$$\|f(x) + \frac{a\tau}{2}{}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha} f(x)\|_{L^2(\Omega)} \leq \|f(x)\|_{L^2(\Omega)}.$$

Proof. By Property 2.1 and notice that $0.5 < \alpha < 1$, we find that the term $\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} f(x), {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} f(x)\right)$ is negative. Thus, for τ sufficiently small

$$\begin{aligned}
& \|f(x) + \frac{a\tau}{2} {}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha} f(x)\|_{L^2(\Omega)}^2 \\
= & \left(f(x) + \frac{a\tau}{2} {}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha} f(x), f(x) + \frac{a\tau}{2} {}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha} f(x)\right) \\
\leq & \left(f(x), f(x)\right) + a\tau \left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} f(x), {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} f(x)\right) + O(\tau^2) \\
\leq & \left(f(x), f(x)\right).
\end{aligned}$$

Next let's consider the stability of the semidiscrete problem.

Theorem 4.1 *For τ sufficiently small , the Problem (9) is stable, and it holds*

$$\|u^{i+1}\|_{J_C^\alpha(\Omega)} \leq \|u^0\|_{L^2(\Omega)} + T\|f\|_{L^\infty(I;L^2(\Omega))}, \ \ i = 0, 1, \cdots, M-1.$$

Proof. For $i = 0$, Problem (9) reads

$$(u^1, v) - \frac{a\tau}{2}\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} u^1, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} v\right) = (u^0, v) + \frac{a\tau}{2}\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} u^0, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} v\right) + \tau\left(f^{\frac{1}{2}}, v\right), \ \forall v \in H_0^\alpha(\Omega)$$

Taking $v = u^1$, then for the left-hand side

$$\|u^1\|_{J_C^\alpha(\Omega)}^2 = (u^1, u^1) - \frac{a\tau}{2}\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} u^1, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} u^1\right).$$

For the right-hand side, according to Lemma 4.1, we have

$$\begin{aligned}
& |(u^0, u^1) + \frac{a\tau}{2}\left({}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} u^0, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} u^1\right) + \tau\left(f^{\frac{1}{2}}, u^1\right)| \\
= & |\left(u^0 + \frac{a\tau}{2} {}_{-1}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha} u^0, u^1\right) + \tau\left(f^{\frac{1}{2}}, u^1\right)| \\
\leq & \|u^0\|_{L^2(\Omega)}\|u^1\|_{L^2(\Omega)} + \tau\|f^{\frac{1}{2}}\|_{L^2(\Omega)}\|u^1\|_{L^2(\Omega)} \\
\leq & \|u^0\|_{L^2(\Omega)}\|u^1\|_{J_C^\alpha(\Omega)} + \tau\|f^{\frac{1}{2}}\|_{L^2(\Omega)}\|u^1\|_{J_C^\alpha(\Omega)}.
\end{aligned}$$

Together above estimates show

$$\|u^1\|_{J_C^\alpha(\Omega)} \leq \|u^0\|_{L^2(\Omega)} + \tau\|f^{\frac{1}{2}}\|_{L^2(\Omega)}.$$

Furthermore, when $i \geq 1$, it can be easily checked that

$$\|u^{i+1}\|_{J_C^\alpha(\Omega)} \leq \|u^i\|_{L^2(\Omega)} + \tau\|f^{i+\frac{1}{2}}\|_{L^2(\Omega)}.$$

By the recurrence relation, we obtain

$$\|u^{i+1}\|_{J_C^\alpha(\Omega)} \leq \|u^0\|_{L^2(\Omega)} + \tau\sum_{j=0}^{i}\|f^{i+\frac{1}{2}}\|_{L^2(\Omega)}.$$

Thus, the following result holds

$$\|u^{i+1}\|_{J_C^\alpha(\Omega)} \leq \|u^0\|_{L^2(\Omega)} + T\|f\|_{L^\infty(I;L^2(\Omega))},$$

which completes the proof. $\square$

Now we turn to prove the following error estimate between the solutions of the semidiscrete and continuous problems.

Theorem 4.2 *Let $u(x,t)$ and $\{u^i\}_{i=0}^M$ the solutions of Problems (5) and (9) respectively, then the following error estimate holds*

$$\|e^{i+1}\|_{J_C^\alpha(\Omega)} \le cT\tau^2,$$

where c is a constant, and $e^{i+1} = u(x,t^{i+1}) - u^{i+1}$, $i = 0,1,\cdots,M-1$.

Proof. When $i=0$, combining (5) and (9) leads to

$$(e^1,v) - \frac{a\tau}{2}\left({}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^\alpha_{\mathbf{x}} e^1, {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^\alpha_{\mathbf{1}} v\right) = (e^0,v) + \frac{a\tau}{2}\left({}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^\alpha_{\mathbf{x}} e^0, {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^\alpha_{\mathbf{1}} v\right) + \tau^3(R^0(x),v).$$

Taking $v = e^1$ and noticing $e^0 = 0$, then

$$\|e^1\|^2_{J_C^\alpha(\Omega)} \le c\tau^3\|e^1\|_{L^2(\Omega)} \le c\tau^3\|e^1\|_{J_C^\alpha(\Omega)},$$

where c is a constant.

Thus we have

$$\|e^1\|_{J_C^\alpha(\Omega)} \le c\tau^3.$$

Assume we have proven $\|e^i\|_{J_C^\alpha(\Omega)} \le ci\tau^3 (i \ge 1)$, then

$$(e^{i+1},v) - \frac{a\tau}{2}\left({}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^\alpha_{\mathbf{x}} e^{i+1}, {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^\alpha_{\mathbf{1}} v\right) = (e^i,v) + \frac{a\tau}{2}\left({}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^\alpha_{\mathbf{x}} e^i, {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^\alpha_{\mathbf{1}} v\right) + \tau^3(R^i(x),v).$$

Taking $v = e^{i+1}$ and using Lemma 4.1, we have

$$\begin{aligned}\|e^{i+1}\|^2_{J_C^\alpha(\Omega)} &\le \|e^i\|_{L^2(\Omega)}\|e^{i+1}\|_{L^2(\Omega)} + c\tau^3\|e^{i+1}\|_{L^2(\Omega)} \\ &\le \|e^i\|_{L^2(\Omega)}\|e^{i+1}\|_{J_C^\alpha(\Omega)} + c\tau^3\|e^{i+1}\|_{J_C^\alpha(\Omega)}.\end{aligned}$$

Then we conclude

$$\|e^{i+1}\|_{J_C^\alpha(\Omega)} \le \|e^i\|_{J_C^\alpha(\Omega)} + c\tau^3 \le ci\tau^3 + c\tau^3 = c(i+1)\tau^3 \le cT\tau^2. \quad \square$$

Theorems 4.1 and 4.2 guarantee we can carry out full discretization for semidiscrete Problem (9) in next section.

5 Full discretization

As is known, the Galerkin spectral discretization proceeds by approximating the solution by the polynomials of high degree. Thus, we introduce finite dimensional space $P_N^0(\Omega) = P_N(\Omega)\bigcap H_0^\alpha(\Omega)$ where P_N is the polynomial space in which the polynomial degree is less than or equal to N.

It's well-known that the following optimal error estimate [15] for the interpolation polynomials based on the Gauss-Lobatto points holds

$$\|u^{i+1} - I_N u^{i+1}\|_{H^\mu(\Omega)} \lesssim N^{\mu-m}\|u^{i+1}\|_{H^m(\Omega)}, \text{ for } u^{i+1} \in H^m(\Omega),\ 0 \le \mu \le m. \tag{12}$$

Let $\widetilde{u}_N^{i+1}$ be the solution of the finite-dimensional variational problem, i.e., find a $\widetilde{u}_N^{i+1} \in P_N^0(\Omega)$ such that

$$B(\widetilde{u}_N^{i+1}, v_N) = F(v_N),\ \forall\, v_N \in P_N^0(\Omega). \tag{13}$$

Theorem 5.1 *Let u^{i+1} be the solution of variational problem (11). The variational problem (13) admits a unique solution and satisfies following inequality*

$$\|u^{i+1}-\widetilde{u}_N^{i+1}\|_{J_C^\alpha(\Omega)} \lesssim N^{-m}\|u^{i+1}\|_{H^m(\Omega)}+\sqrt{\tau}N^{\alpha-m}\|u^{i+1}\|_{H^m(\Omega)}.$$

Proof. Since $P_N^0(\Omega)$ is a subspace of $H_0^\alpha(\Omega)$ and Problem (11) is well-posed, thus Problem (13) also admits a unique solution. According to Ceá's lemma, it deduces that

$$\begin{aligned}\|u^{i+1}-\widetilde{u}_N^{i+1}\|_{J_C^\alpha(\Omega)} &\lesssim \inf_{v_N\in P_N^0(\Omega)}\|u^{i+1}-v_N\|_{J_C^\alpha(\Omega)} \lesssim \|u^{i+1}-I_Nu^{i+1}\|_{J_C^\alpha(\Omega)}\\ &\lesssim N^{-m}\|u^{i+1}\|_{H^m(\Omega)}+\sqrt{\tau}N^{\alpha-m}\|u^{i+1}\|_{H^m(\Omega)}.\end{aligned}$$

This completes the proof. $\square$

In the following, we will use duality method to obtain the error estimates in L^2-norm. First, two regularity estimates are discussed which will be used for forthcoming analysis.

Lemma 5.1 *Assume that u^{i+1} is the solution of Eq.(6) with its boundary conditions, then for g defined in (10), following inequalities hold*

$$\begin{aligned}\tau|u^{i+1}|_{J_C^{2\alpha}(\Omega)} &\lesssim \|g\|_{L^2(\Omega)},\ \alpha\neq\frac{3}{4},\\ \tau|u^{i+1}|_{J_C^{2\alpha-\varepsilon}(\Omega)} &\lesssim \|g\|_{L^2(\Omega)},\ \alpha=\frac{3}{4},\ 0<\varepsilon<\frac{3}{8}.\end{aligned}$$

Proof. Rewrite Eq.(6) as

$$u^{i+1}-\frac{a\tau}{2}{}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha}u^{i+1}=g.$$

We multiply the both sides of above equation by ${}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha}u^{i+1}$ and integrate over Ω, then have

$$\frac{a\tau}{2}\left({}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha}u^{i+1},{}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha}u^{i+1}\right)=\left(u^{i+1},{}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha}u^{i+1}\right)-\left(g,{}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha}u^{i+1}\right).$$

For the left-hand side, by Lemma 2.2, it deduces that

$$|\left({}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha}u^{i+1},{}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha}u^{i+1}\right)|\gtrsim |u^{i+1}|^2_{J_L^{2\alpha}(\Omega)}.$$

For the right-hand side, one can get

$$\begin{aligned}|\left(u^{i+1},{}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha}u^{i+1}\right)| &\leq \|u^{i+1}\|_{L^2(\Omega)}|u^{i+1}|_{J_L^{2\alpha}(\Omega)},\\ |\left(-g,{}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}_{\mathbf{x}}^{\mathbf{2}\alpha}u^{i+1}\right)| &\leq \|g\|_{L^2(\Omega)}|u^{i+1}|_{J_L^{2\alpha}(\Omega)}.\end{aligned}$$

Combining the estimates of both sides, we have

$$\tau|u^{i+1}|^2_{J_L^{2\alpha}(\Omega)}\lesssim \|u^{i+1}\|_{L^2(\Omega)}|u^{i+1}|_{J_L^{2\alpha}(\Omega)}+\|g\|_{L^2(\Omega)}|u^{i+1}|_{J_L^{2\alpha}(\Omega)}.$$

Furthermore,

$$\tau|u^{i+1}|_{J_L^{2\alpha}(\Omega)}\lesssim \|u^{i+1}\|_{L^2(\Omega)}+\|g\|_{L^2(\Omega)}.$$

Notice that $\|u^{i+1}\|_{L^2(\Omega)}\lesssim\|g\|_{L^2(\Omega)}$, then

$$\tau|u^{i+1}|_{J_C^{2\alpha}(\Omega)}\lesssim\|g\|_{L^2(\Omega)},$$

which completes the proof for $\alpha\neq\frac{3}{4}$. For $\alpha=\frac{3}{4}$ and $0<\varepsilon<\frac{3}{8}$, by Property 2.2, it holds

$$|u^{i+1}|_{J_C^{2\alpha-\varepsilon}(\Omega)}\lesssim|u^{i+1}|_{J_L^{2\alpha}(\Omega)}.$$

Then we can follow the similar way to prove the result. $\square$

Theorem 5.2 *Let $u^{i+1} \in H_0^{\alpha}(\Omega)\bigcap H^m(\Omega)$ $(\alpha \le m)$ and $\widetilde{u}_N^{i+1}$ solve Problems (11) and (13) respectively, then if time step size τ and polynomial degree N satisfy $\tau = O(N^{-\alpha})$, we have*

$$\begin{aligned}\|u^{i+1} - \widetilde{u}_N^{i+1}\|_{L^2(\Omega)} &\lesssim \tau N^{\alpha-m}\|u^{i+1}\|_{H^m(\Omega)},\ \alpha \ne \frac{3}{4},\\ \|u^{i+1} - \widetilde{u}_N^{i+1}\|_{L^2(\Omega)} &\lesssim \tau N^{\alpha+\varepsilon-m}\|u^{i+1}\|_{H^m(\Omega)},\ \alpha = \frac{3}{4},\ 0<\varepsilon<\frac{3}{8}.\end{aligned}$$

Proof. Introduce the adjoint problem: find a $\omega \in H_0^{\alpha}(\Omega)$ such that

$$B(\omega, v) = (u^{i+1} - \widetilde{u}_N^{i+1}, v),\ \forall\ v \in H_0^{\alpha}(\Omega).$$

Now we deduce the case of $\alpha \ne \frac{3}{4}$, the case for $\alpha = \frac{3}{4}$ is similar. By means of Lemma 5.1, ω satisfies the regularity estimate

$$\tau|u^{i+1}|_{J_C^{2\alpha}(\Omega)} \lesssim \|u^{i+1} - \widetilde{u}_N^{i+1}\|_{L^2(\Omega)}.$$

Taking $v = u^{i+1} - \widetilde{u}_N^{i+1}$ and using the Galerkin orthogonality, we have

$$\begin{aligned}\|u^{i+1} - \widetilde{u}_N^{i+1}\|^2_{L^2(\Omega)} &= B(\omega, u^{i+1} - \widetilde{u}_N^{i+1}) = B(\omega - I_N\omega, u^{i+1} - \widetilde{u}_N^{i+1})\\ &\lesssim \|\omega - I_N\omega\|_{J_C^{\alpha}(\Omega)}\|u^{i+1} - \widetilde{u}_N^{i+1}\|_{J_C^{\alpha}(\Omega)}\\ &\lesssim \sqrt{N^{-4\alpha} + \tau N^{-2\alpha}}|\omega|_{J_C^{2\alpha}(\Omega)}\|u^{i+1} - \widetilde{u}_N^{i+1}\|_{J_C^{\alpha}(\Omega)}\\ &\lesssim \tau^{3/2}|\omega|_{J_C^{2\alpha}(\Omega)}\|u^{i+1} - \widetilde{u}_N^{i+1}\|_{J_C^{\alpha}(\Omega)}\\ &\lesssim \sqrt{\tau}\|u^{i+1} - \widetilde{u}_N^{i+1}\|_{L^2(\Omega)}\|u^{i+1} - \widetilde{u}_N^{i+1}\|_{J_C^{\alpha}(\Omega)}.\end{aligned}$$

Thus according to Theorem 5.1, we conclude

$$\|u^{i+1} - \widetilde{u}_N^{i+1}\|_{L^2(\Omega)} \lesssim \sqrt{\tau}\|u^{i+1} - \widetilde{u}_N^{i+1}\|_{J_C^{\alpha}(\Omega)} \lesssim \tau N^{\alpha-m}\|u^{i+1}\|_{H^m(\Omega)}.\quad \square$$

Let u_N^i be the numerical solution at time t_i of the full discretization, namely $\{u_N^i\}_{i=1}^M$ satisfies

$$\begin{aligned}&(u_N^{i+1}, v_N) - \frac{a\tau}{2}\left({}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\alpha}_{\mathbf{x}}u_N^{i+1}, {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\alpha}_{\mathbf{1}}v_N\right)\\ =\ &(u_N^{i}, v_N) + \frac{a\tau}{2}\left({}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\alpha}_{\mathbf{x}}u_N^{i}, {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\alpha}_{\mathbf{1}}v_N\right) + \tau(f^{i+\frac{1}{2}}, v_N),\ \forall\ v_N \in P_N^0(\Omega).\end{aligned}\tag{14}$$

Now we deduce the error estimates for $u(x,t_i) - u_N^i$ in two different norms $\|\cdot\|_{J_C^{\alpha}(\Omega)}$ and $\|\cdot\|_{L^2(\Omega)}$.

Theorem 5.3 *Let $u(x,t)$ be the exact solution of Problem (2), and $\{u_N^i\}_{i=1}^M$ solves Problem (14). If $u \in H^1(I; H_0^{\alpha}(\Omega)\bigcap H^m(\Omega))$, then following error estimate holds*

$$\|u(\cdot,t_i) - u_N^i\|_{J_C^{\alpha}(\Omega)} \lesssim T(\tau^2 + \tau^{-1}\max\{N^{-\alpha}, \sqrt{\tau}\}N^{\alpha-m})\|u\|_{L^{\infty}(I;H^m(\Omega))},\ i = 1,2,\cdots,M.$$

Proof. We know that $\{u(x,t_i)\}_{i=1}^M$ satisfy

$$\begin{aligned}&(u(\cdot,t_{i+1}), v) - \frac{a\tau}{2}\left({}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\alpha}_{\mathbf{x}}u(\cdot,t_{i+1}), {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\alpha}_{\mathbf{1}}v\right)\\ =\ &(u(\cdot,t_i), v) + \frac{a\tau}{2}\left({}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\alpha}_{\mathbf{x}}u(\cdot,t_i), {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\alpha}_{\mathbf{1}}v\right) + \tau(f^{i+\frac{1}{2}}, v) + (R^i(\cdot)\tau^3, v),\ \forall\ v \in H_0^{\alpha}(\Omega).\end{aligned}$$

Let Π_N^{α} be the orthogonal projection operator from $H_0^{\alpha}(\Omega)$ to $P_N^0(\Omega)$ with respect to $\|\cdot\|_{H^{\alpha}(\Omega)}$, namely

$$\begin{aligned}&(\Pi_N^{\alpha}u(\cdot,t_{i+1}), v_N) - \frac{a\tau}{2}\left({}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\alpha}_{\mathbf{x}}\Pi_N^{\alpha}u(\cdot,t_{i+1}), {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\alpha}_{\mathbf{1}}v_N\right)\\ =\ &(u(\cdot,t_i), v_N) + \frac{a\tau}{2}\left({}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\alpha}_{\mathbf{x}}u(\cdot,t_i), {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\alpha}_{\mathbf{1}}v_N\right) + \tau(f^{i+\frac{1}{2}}, v_N) + (R^i(\cdot)\tau^3, v_N),\ \forall\ v_N \in H_0^{\alpha}(\Omega).\end{aligned}$$

Let $\widetilde{e}_N^{i+1} = \Pi_N^\alpha u(\cdot, t_{i+1}) - u_N^{i+1}$ and $e_N^{i+1} = u(\cdot, t_{i+1}) - u_N^{i+1}$. Subtracting (14) from above equation, it becomes

$$\begin{aligned} & (\widetilde{e}_N^{i+1}, v_N) - \frac{a\tau}{2}\left({}_{\mathbf{-1}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha}\widetilde{e}_N^{i+1}, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} v_N\right) \\ = \;& (e_N^i, v_N) + \frac{a\tau}{2}\left({}_{\mathbf{-1}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{x}}^{\alpha} e_N^i, {}_{\mathbf{x}}^{\mathbf{RL}}\mathbf{D}_{\mathbf{1}}^{\alpha} v_N\right) + (R^i(\cdot)\tau^3, v_N), \ \forall\, v_N \in H_0^\alpha(\Omega). \end{aligned}$$

Taking $v_N = \widetilde{e}_N^{i+1}$ and by Lemma 4.1, we have

$$\|\widetilde{e}_N^{i+1}\|_{J_C^\alpha(\Omega)} \leq \|e_N^i\|_{L^2(\Omega)} + c\tau^3.$$

Using the triangular inequality and Theorem 5.1,

$$\begin{aligned} \|e_N^{i+1}\|_{J_C^\alpha(\Omega)} &\leq \|u(\cdot, t_{i+1}) - \Pi_N^\alpha u(\cdot, t_{i+1})\|_{J_C^\alpha(\Omega)} + \|\widetilde{e}_N^{i+1}\|_{J_C^\alpha(\Omega)} \\ &\lesssim \max\{N^{-\alpha}, \sqrt{\tau}\} N^{\alpha-m} \|u(\cdot, t_{i+1})\|_{H^m(\Omega)} + \|e_N^i\|_{L^2(\Omega)} + \tau^3 \\ &\lesssim (i+1)(\tau^3 + \max\{N^{-\alpha}, \sqrt{\tau}\} N^{\alpha-m} \|u\|_{L^\infty(I;H^m(\Omega))}) \\ &\lesssim T(\tau^2 + \tau^{-1}\max\{N^{-\alpha}, \sqrt{\tau}\} N^{\alpha-m}) \|u\|_{L^\infty(I;H^m(\Omega))}. \end{aligned}$$

The proof is completed. $\square$

Remark 5.1 *From above theorem, the error estimate in* $\|\cdot\|_{L^2(\Omega)}$ *can be roughly obtained as follows*

$$\|u(\cdot, t_i) - u_N^i\|_{L^2(\Omega)} \lesssim T(\tau^2 + \tau^{-1}\max\{N^{-\alpha}, \sqrt{\tau}\} N^{\alpha-m}) \|u\|_{L^\infty(I;H^m(\Omega))}.$$

If $\tau = O(N^{-\alpha})$ *is assumed, then it deduces that*

$$\|u(\cdot, t_i) - u_N^i\|_{L^2(\Omega)} \lesssim T(\tau^2 + N^{3\alpha/2-m}) \|u\|_{L^\infty(I;H^m(\Omega))}.$$

However, using Theorem 5.2, we will obtain a better estimate in norm $\|\cdot\|_{L^2(\Omega)}$.

Theorem 5.4 *Let* $u(x,t)$ *and* $\{u_N^i\}_{i=1}^M$ *be the exact solutions of Problems (2) and (14) respectively. If* $u \in H^1(I; H_0^\alpha(\Omega) \bigcap H^m(\Omega))$ *and* $\tau = O(N^{-\alpha})$, *then the error estimates satisfy*

$$\begin{aligned} \|u(\cdot, t_i) - u_N^i\|_{L^2(\Omega)} &\lesssim T(\tau^2 + N^{\alpha-m}) \|u\|_{L^\infty(I;H^m(\Omega))}, \ \alpha \neq \frac{3}{4}, \\ \|u(\cdot, t_i) - u_N^i\|_{L^2(\Omega)} &\lesssim T(\tau^2 + N^{\alpha+\varepsilon-m}) \|u\|_{L^\infty(I;H^m(\Omega))}, \ \alpha = \frac{3}{4}, \ 0 < \varepsilon < \frac{3}{8}. \end{aligned}$$

Proof. These two inequalities can be proved similarly as Theorem 5.3 by using $\|\widetilde{e}_N^{i+1}\|_{L^2(\Omega)} \leq \|\widetilde{e}_N^{i+1}\|_{J_C^\alpha(\Omega)}$ and Theorem 5.2. $\square$

6 Numerical experiments

In this section, we will verify the theoretical results by doing numerical experiments. Two numerical examples are provided below, one is the one-dimensional space fractional diffusion equation, the other is two-dimensional space fractional one.

Noticing that our problem is equipped with homogeneous boundary conditions, we can express the function u_N^{i+1} in terms of the Lagrangian interpolants based on $N+1$ Gauss-Legendre-Lobatto points, i.e.,

$$u_N^{i+1}(x) = \sum_{k=1}^{N-1} C_k^{i+1} h_k(x),$$

where C_k^{i+1} is the unknown discrete solution of $u_N^{i+1}(x)$ at the $k-$th Gauss-Legendre-Lobatto point; $h_k(x)$ is the $k-$th Lagrangian polynomial with respect to the given Gauss points.

However generally speaking, it's difficult to obtain the explicit expression for the Riemann-Liouville fractional derivative of Lagrangian polynomial $h_k(x)$. To overcome this difficulty and facilitate computing, we establish following Lemma.

Lemma 6.1 *For $\mu > 0$, it holds that*

$$ {}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}^{\mu}_{\mathbf{x}} L_n(x) = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}(1+x)^{-\mu} P_n^{\mu,-\mu}(x), \tag{15} $$

$$ {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\mu}_{\mathbf{1}} L_n(x) = \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}(1-x)^{-\mu} P_n^{-\mu,\mu}(x), \tag{16} $$

where $L_n(x)$ is a Legendre polynomial, $P_n^{\alpha,\beta}(x)$ is a Jacobi polynomial.

Proof. According to [1], we can directly obtain

$$ {}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}^{\mu}_{\mathbf{x}} \left[(1+x)^{\beta+\mu} P_n^{\alpha-\mu,\beta+\mu}(x)\right] = \frac{\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)} \frac{P_n^{\alpha-\mu,\beta+\mu}(-1)}{P_n^{\alpha,\beta}(-1)}(1+x)^{\beta} P_n^{\alpha,\beta}(x), \tag{17} $$

$$ {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\mu}_{\mathbf{1}} \left[(1-x)^{\alpha+\mu} P_n^{\alpha+\mu,\beta-\mu}(x)\right] = \frac{\Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)} \frac{P_n^{\alpha+\mu,\beta-\mu}(1)}{P_n^{\alpha,\beta}(1)}(1-x)^{\alpha} P_n^{\alpha,\beta}(x). \tag{18} $$

Thus it can be checked that for (15) and (16), we only need to set $\alpha = \mu,\ \beta = -\mu$ in (17) and $\alpha = -\mu,\ \beta = \mu$ in (18) respectively. $\square$

Thus after solving this system $h_k(x) = \sum\limits_{n=0}^{N} D_n^k L_n(x)$ and by Lemma 6.1, we have

$$ {}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}^{\alpha}_{\mathbf{x}} h_k(x) = \sum_{n=0}^{N} D_n^k \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}(1+x)^{-\mu} P_n^{\mu,-\mu}(x), \tag{19} $$

$$ {}^{\mathbf{RL}}_{\mathbf{x}}\mathbf{D}^{\alpha}_{\mathbf{1}} h_k(x) = \sum_{n=0}^{N} D_n^k \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}(1-x)^{-\mu} P_n^{-\mu,\mu}(x). \tag{20} $$

6.1 One-dimensional problem

Consider a one-dimensional space fractional diffusion equation

$$ \begin{cases} \dfrac{\partial u(x,t)}{\partial t} - a\,{}^{\mathbf{RL}}_{-\mathbf{1}}\mathbf{D}^{\mathbf{2}\alpha}_{\mathbf{x}} u(x,t) = f(x,t),\ x \in \Omega,\ t \in I, \\ u(x,0) = e^{-40x^2},\ x \in \Omega, \\ u(-1,t) = u(1,t) = cos(t)e^{-40},\ t \in I, \end{cases} $$

with $a = 0.5$, $\Omega = (-1, 1)$, and its exact solution is

$$ u(x,t) = cos(t)e^{-40x^2}. $$

We compute the figure and the errors $\|u(\cdot,T) - u_N^M\|_{L^2(\Omega)}$ at time $T = 1$ with $\alpha = 0.8$. The comparisons of numerical solution and exact solution is shown in Figure 1. For this case, we take time step size $\tau = 0.2$ and the degree of interpolation polynomial in spatial direction $N = 20$. It can be seen that our numerical results are in excellent agreement with the exact solution. In Table 1, we take $N = 40$, a value large enough such that the spatial discretization errors are negligible as compared with the time errors, and choose different time step size to obtain the numerical convergence order in time. We can check that these numerical convergence order, almost approach 2, are consistent with the theoretical analysis in Section 5. In Figure 2, with sufficiently small $\tau = 0.001$, we plot the errors as functions of polynomial degree N. As expected, the errors show an exponential decay.

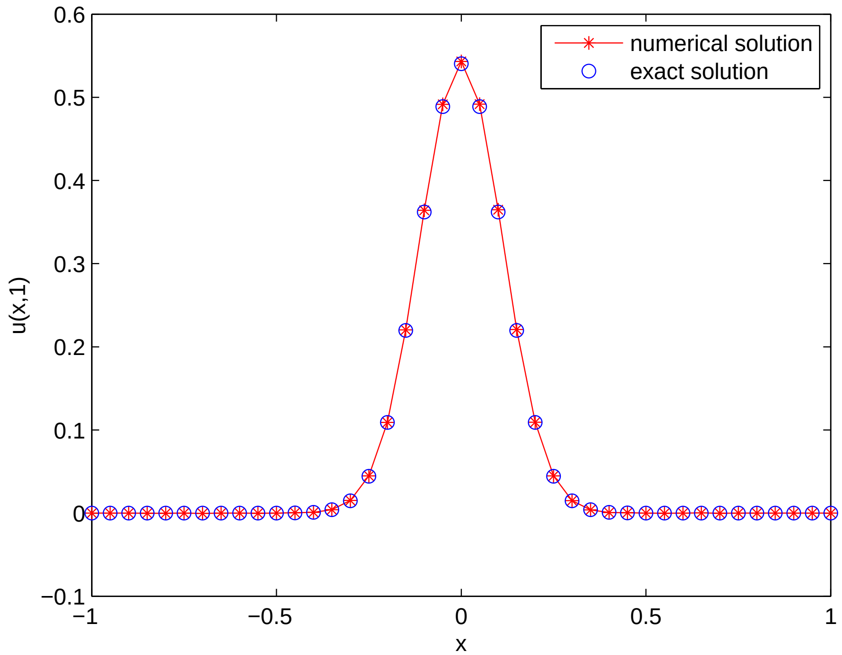

Figure 1: The comparisons of numerical solution and exact solution for $\alpha = 0.8$, $\tau = 0.2$ and $N = 30$.

Table 1: The errors for different step size τ and $\alpha = 0.8$, $N = 40$.

step size τ	$\left\Vert u(x,1) - u_N^M \right\Vert_{L^2(\Omega)}$	convergence order
1/5	3.069836e-3	
1/10	7.004634e-4	2.1318
1/20	1.697978e-4	2.0445
1/40	4.249987e-5	1.9983
1/80	1.068166e-5	1.9923

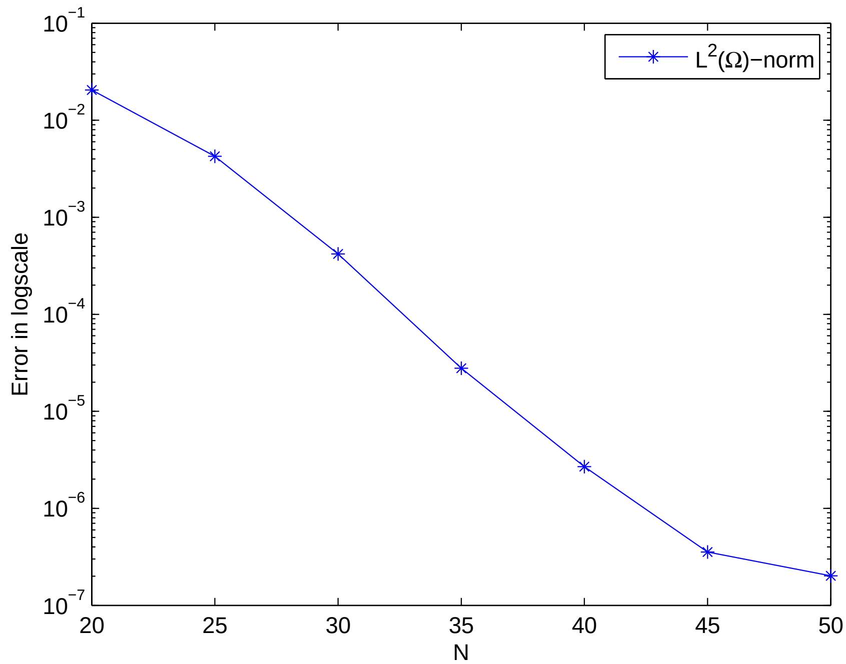

Figure 2: For $\alpha = 0.8$ and $\tau = 0.001$, the errors show an exponential decay with N increased.

6.2 Two-dimensional problem

Now let's consider a two-dimensional fractional diffusion equation

$$\begin{cases} \dfrac{\partial u(x,y,t)}{\partial t} - {}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\mathbf{2}\alpha}_{\mathbf{x}} u(x,y,t) - {}^{\mathbf{RL}}_{\mathbf{-1}}\mathbf{D}^{\mathbf{2}\beta}_{\mathbf{y}} u(x,y,t) = f(x,y,t),\ (x,y) \in \Omega,\ t \in I, \\ u(x,y,0) = sin(\frac{\pi}{2}x + \frac{\pi}{2})sin(\frac{\pi}{2}y + \frac{\pi}{2}),\ (x,y) \in \Omega, \\ u(x,y,t) = 0,\ (x,y) \in \partial\Omega, \end{cases}$$

with $0.5 < \alpha, \beta < 1$, $\Omega = (-1,1) \times (-1,1)$, and $I = (0,1]$. Numerical experiments are carried out with the exact analytical solution

$$u(x,y,t) = cos(t)sin(\frac{\pi}{2}x + \frac{\pi}{2})sin(\frac{\pi}{2}y + \frac{\pi}{2}).$$

In Table 2, we take $N_x = 20$ and $N_y = 20$, the polynomial degree in x-direction and in y-direction respectively, to make the error in space sufficient small. Under this conditions, Table 2 also explains that the convergence order in time direction is close to 2. In Figure 3, taking $\tau = 0.001$, $\beta = 0.75$ and $N_y = 20$, we can observe the spectral convergence in x-direction. Very similar figure can be obtained to demonstrate the convergence in y-direction for $\tau = 0.001$, $\alpha = 0.75$ and $N_x = 20$.

step size τ	$\left\|u(x,y,1)-u^M_{N_x,N_y}\right\|_{L^2(\Omega)}$	convergence order
1/5	3.311963e-3	
1/10	8.235122e-4	2.0078
1/20	2.055152e-4	2.0025
1/40	5.135596e-5	2.0006
1/80	1.283756e-5	2.0002

Table 2: The errors for different step size τ and $\alpha = 0.75$, $\beta = 0.75$, $N_x = 20$ and $N_y = 20$.

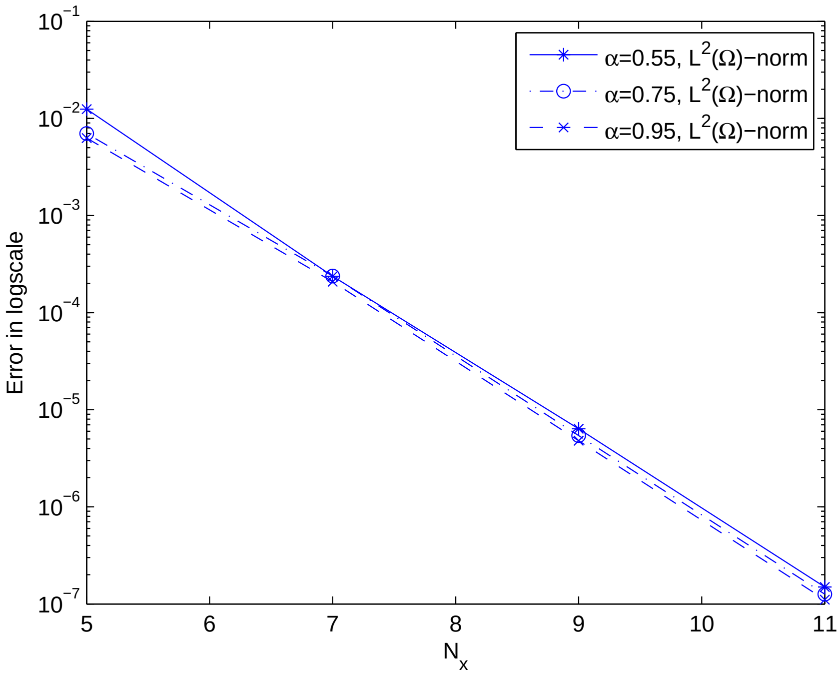

Figure 3: For $\tau = 0.001$, $\beta = 0.75$, $N_y = 20$ and different α, the errors show an exponential decay with N_x increased.

7 Concluding remarks

In this paper, we have proposed a new high order numerical method for the space fractional diffusion equation with convergence order $O(\tau^2+N^{\alpha-m})$ in L^2-norm by combining the classical Crank-Nicholson method and the spectral Galerkin method, and have rigorously proved the stability and convergence of this method. So far, it seems that no other published numerical method for space fractional diffusion equation can achieve so high accuracy in time and in space simultaneously. It should be mentioned that our method is also valid for solving two-dimensional problem, or even higher dimensional case. The numerical examples have verified the theoretical results. It is demonstrated that this method is an effective and high accuracy numerical scheme for solving a kind of space fractional diffusion equations.

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