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Long-time behavior for Navier–Stokes flows in a two-dimensional exterior domain

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ABSTRACT

There are several long standing problems on the incompressible Navier–Stokes flows in 2D exterior domains, which claim how to characterize L^1-summability of the 2D N-S flows; whether the total net force exerted on the boundary is finite; and how to establish decay results of higher-order spatial derivatives, including the weighted cases. In order to solve these questions, we firstly find some types of new technical inequalities, which are used to overcome the difficulties caused by the domain boundary; using $L^q - L^r$ properties for non-stationary Stokes flows, together with elliptic estimates for the steady Stokes system, we can avoid the strong singularity and answer these mentioned problems completely. It should be pointed out that main results in this article are motivated by the works in [5,37], respectively.

1. Introduction and main results

Consider a moving body in a viscous incompressible fluid filling the whole space $\mathbb{R}^2$. If we describe the fluid motion by using a coordinate system attached to the body, we

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obtain the exterior problem for the Navier–Stokes equations with homogeneous boundary conditions:

$$\begin{cases} \partial_t u - \Delta u + (u\cdot\nabla)u + \nabla p = 0 & \text{in } \ \Omega\times(0,\infty), \\ \nabla\cdot u = 0 & \text{in } \ \Omega\times(0,\infty), \\ u(x,t) = 0 & \text{on } \ \partial\Omega\times(0,\infty), \\ u(x,t) \longrightarrow 0 & \text{as } \ |x| \longrightarrow \infty, \\ u(x,0) = a & \text{in } \ \Omega, \end{cases} \tag{1.1}$$

where Ω is an exterior domain (the complement $\mathbb{R}^2\backslash\Omega$ is compact) of $\mathbb{R}^2$, of class $C^{2+\mu}$ $(0 < \mu < 1)$. In general we shall need some kind of smoothness property for a domain $Q \subset \mathbb{R}^2$, and assume that Q is of class C^m-smoothness with $m \geq 1$ (which must be specified) in the sense: the boundary ∂Q is a 1-dimensional manifold of class C^m and Q is locally located on one side of the boundary ∂Q.

Without loss of generalization, we always assume $0 \notin \overline{\Omega}$. In problem (1.1), $u = u(x,t) = (u_1(x,t), u_2(x,t))$ and $p = p(x,t)$ denote unknown velocity vector and the pressure respectively, while initial data $a(x)$ is assumed to satisfy a *compatibility condition*: $\nabla\cdot a = 0$ in Ω and the normal component of a equals to zero on $\partial\Omega$; and

$$\partial_t = \frac{\partial}{\partial t}, \quad \nabla = (\partial_1, \partial_2), \quad \partial_j = \frac{\partial}{\partial x_j}\ (j = 1, 2),$$

$$\Delta u = \sum_{j=1}^{2} \partial_j\partial_j u, \quad (u\cdot\nabla)u = \sum_{j=1}^{2} u_j\partial_j u, \quad \nabla\cdot u = \sum_{j=1}^{2} \partial_j u_j.$$

Before stating our results, we introduce some function spaces and then give the definition of weak solution. In this article, $L^q(\Omega)$ with $1 \leq q \leq \infty$ stands for the usual Lebesgue space of vector-valued functions; $C^\infty_{0,\sigma}(\Omega)$ denotes the set of all C^∞ real vector-valued functions $\phi = (\phi_1, \phi_2)$ with compact support in Ω, and $\nabla\cdot\phi = 0$ in Ω. Let $L^r_\sigma(\Omega)$ with $1 < r < \infty$ be the closure of $C^\infty_{0,\sigma}(\Omega)$ with respect to $\|\cdot\|_{L^r(\Omega)}$, which can also be denoted in another way (see [40]):

$$L^r_\sigma(\Omega) = \{v \in L^r(\Omega);\ \ \nabla\cdot v = 0,\ v\cdot\nu\,|_{\partial\Omega} = 0\}.$$

Moreover, the Helmholtz decomposition holds (see [23]):

$$L^r(\Omega) = L^r_\sigma(\Omega) \oplus L^r_\pi(\Omega),$$

where

$$L^r_\pi(\Omega) = \{w = \nabla p \in L^r(\Omega);\ \ p \in L^r_{loc}(\overline{\Omega})\}.$$

Given $a \in L^2_\sigma(\Omega)$, a vector function $u \in L^\infty(0,\infty; L^2_\sigma(\Omega)) \bigcap L^2_{loc}([0,\infty); W^{1,2}_0(\Omega))$ is called a weak solution of (1.1) if u satisfies (1.1) in the sense of distribution. It is well known that a global-in-time weak solution of (1.1) exists uniquely for all $a \in L^2_\sigma(\Omega)$ and is classical on $\Omega \times (0, \infty)$ (see [40] for example).

The problem of describing the dynamics of an incompressible fluid, due to its significance for both theory and application, has attracted the attention of many researchers. The core is about the existence of a global large smooth solution to the Navier–Stokes equations. In his famous paper, Leray [28] constructed a global weak solution in the three dimensional energy space for Navier–Stokes equations, and the uniqueness of these solutions is only known in space dimension two. Meanwhile it is also well known that smooth solutions are global in dimension two and for higher dimensions when the data are small in some critical spaces, see Ladyzhenskaya's book [27], where a sufficiently full analysis of the current state of the problem and an overview of existing literature and proposed solution methods are provided. In the three dimensional case, a large gap remains between the regularity available in the existence results and additional regularity required in the sufficient conditions to guarantee the smoothness of weak solutions. Substantial results have been narrowed and arrived at in a series of excellent works of Caffarelli, Kohn and Nirenberg [16], Lin [29], Iskauriaza, Seregin and Sverak [26], Serrin [39], which bring about a deeper understanding on the regularity of the Navier Stokes system. Of course, this list is far from complete, and for further results, it is referred to [2,18,19,38] and the references therein.

Let $a \in L^2_\sigma(\Omega)$, and let u be the (strong) solution of (1.1). Borchers and Miyakawa [10] obtained the temporal L^2 estimate of the Navier–Stokes solution u in 2D exterior domains:

$$\|u(t)\|_{L^2(\Omega)} \leq Ct^{-(\frac{1}{r}-\frac{1}{2})}, \quad 1 < r \leq 2$$

if the initial velocity $a \in L^r(\Omega) \cap L^2_\sigma(\Omega)$. Subsequently, the decay rates for the 2D Navier–Stokes flow u are shown by Bae and Jin [6], that is, if $a \in L^r(\Omega)$ with $1 < r \leq q < \infty$ or $1 < r < q = \infty$, then

$$\|u(t)\|_{L^q(\Omega)} = O(t^{-\frac{1}{r}+\frac{1}{q}}) \quad \text{as} \quad t \longrightarrow \infty.$$

Moreover

$$\|\nabla u(t)\|_{L^q(\Omega)} = O(t^{-\frac{1}{r}+\frac{1}{q}-\frac{1}{2}}), \quad 1 < r \leq q \leq 2 \quad \text{as} \quad t \longrightarrow \infty.$$

In addition, if $|x|a \in L^{\frac{2r}{2-r}}(\Omega)$ with $1 < r \leq \frac{2q}{q+2} < 2 \leq q < \infty$, and if $0 < \alpha \leq 1$, then for any small $\delta > 0$

$$\|(1 + |x|)^\alpha u(t)\|_{L^q(\Omega)} = O(t^{-\frac{1}{2}+\frac{1}{q}+\frac{\alpha}{2}+\delta}) \quad \text{as} \quad t \longrightarrow \infty.$$

Recently, under some restrictive conditions on initial data, Han [20] improved this weighted result:

$$\| |x|^\alpha u(t) \|_{L^q(\Omega)} = O\big(t^{\frac{\alpha}{2}-(1-\frac{1}{q})} \log_e(1+t)\big) \quad \text{as} \quad t \longrightarrow \infty,$$

provided $1 < q < \infty$ and $0 < \alpha < 1$.

He and Miyakawa [23] obtained faster decay rates in $L^q(\Omega)$ $(1 \le q \le \infty)$ for the solution u of (1.1) under some appropriate conditions, but it is difficult to check if the solution u of (1.1) satisfies such restrictive conditions or not. In addition, Constantin and Wu [17] focused on the decay properties for a class of quasi-geostrophic equations. There is an extensive literature on the decay properties for weak (or strong) solutions of (1.1) in n-dimensional cases. The readers are referred to [1,3–9,12,13,15,22,24,31–34] and the references therein.

Theorem 1.1. *Let $a \in L^2_\sigma(\Omega)$. Then problem (1.1) possesses a unique strong solution (u,p). If $a \in L^1(\Omega) \bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1 < q \le 2$, the solution (u,p) satisfies for $t \ge T_0$ with some number $T_0 > 1$*

$$\|\nabla u(t)\|_{L^r(\Omega)} \le Ct^{-\frac{1}{2}-(1-\frac{1}{r})}, \quad \|\partial_t u(t)\|_{L^r(\Omega)} \le Ct^{-1-(1-\frac{1}{r})}, \quad 1 \le r \le \infty; \tag{1.2}$$

$$\|\nabla^2 u(t)\|_{L^r(\Omega)} \le \begin{cases} Ct^{-1-(1-\frac{1}{r})} & \text{if } 1 \le r < \infty, \\ Ct^{-\frac{7}{4}} & \text{if} \quad r = \infty. \end{cases} \tag{1.3}$$

In addition, assume $|x|^{1+\alpha} a \in L^1(\Omega)$, $|x|^\alpha a \in L^2(\Omega)$ with $0 \le \alpha \le 1$, then for $1 \le r \le \infty$, $(\alpha, r) \ne (1,1)$ and $t \ge T_0$

$$\Big\| |x|^\alpha \big(u(t) - V(\cdot,t) \cdot \int_0^t \mathcal{F}(\tau)d\tau\big) \Big\|_{L^r(\Omega)} \le C(t^{-1} + t^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})} \log_e t). \tag{1.4}$$

In particular, it holds for $0 \le \alpha < 1$ and $t \ge T_0$

$$\Big\| |x|^\alpha \big(u(t) - V(\cdot,t) \cdot \int_0^t \mathcal{F}(\tau)d\tau\big) \Big\|_{L^1(\Omega)} \le Ct^{-\frac{1-\alpha}{2}} \log_e t.$$

Here $e = 2.71828\ldots$ is the natural number, and the 2×2 matrix function $V(x,t)$ is given as follows:

$$V(x,t) = (V_{j,k}(x,t))_{j,k=1,2} \quad \text{and} \quad V_{j,k}(x,t) = E_t(x)\delta_{j,k} + \int_0^\infty \partial_j \partial_k E_{t+\tau}(x) d\tau,$$

$E_t(x) = (4\pi t)^{-1} e^{-\frac{|x|^2}{4t}}$ is the Gaussian kernel.

$$\mathcal{F}(t) = \int_{\partial\Omega} (T[u,p] \cdot \nu)(y,t) dS_y$$

denotes the net force to the boundary $\partial\Omega$*, where* $T[u,p] = (T_{jk}[u,p])_{j,k=1}^2$*,* $T_{jk}[u,p] = \partial_j u_k + \partial_k u_j - \delta_{jk} p$ *is the stress tensor associated to* (u,p)*,* $\nu = (\nu_1, \nu_2)$ *is the unit outward normal to* $\partial\Omega$*.*

Remark. (1) Compared with the result in [20]: $\||x|^\alpha u(t)\|_{L^q(\Omega)} \le Ct^{\frac{\alpha}{2}-(1-\frac{1}{q})}$, $1 < q < \infty$, the estimate (1.4) reveals that the net force to the boundary $\partial\Omega$: $\int_0^t \mathcal{F}(\tau)d\tau$ plays an important role in determining the algebraic decay rates. Especially, the associated net force to the boundary may not vanish in Theorem 1.1.

(2) Let $P: \ L^r(\Omega) \to L^r_\sigma(\Omega)$, $1 < r < \infty$, be the bounded projection, and $A = -P\Delta$ the Stokes operator. It is easy to verify by using the Fourier transform that $P\Delta = \Delta P$ in $\mathbb{R}^2$ because there is no boundary condition. However the situation becomes extremely complicated if the domain boundary is nonempty, and many properties for Stokes operator A in the whole space $\mathbb{R}^2$ are not valid any more for some domains $Q \ne \mathbb{R}^2$. For example (see [11]), let $w \in D(A^{\frac{1}{2}}) = W^{1,r}_{0,\sigma}(Q)$ with $1 < r < \infty$, then $\|\nabla w\|_{L^r(Q)} \le C\|A^{\frac{1}{2}} w\|_{L^r(Q)}$ is valid for $Q = \mathbb{R}^2$ and $1 < r < \infty$; valid for exterior domain Q and $1 < r < 2$, not true if $2 \le r < \infty$. In addition, the following estimate is generally not true,

$$\|\nabla^2 u(t)\|_{L^r(\Omega)} \le C\|Au(t)\|_{L^r(\Omega)} \quad \text{for} \quad 1 \le r < \infty.$$

Whence the estimate (1.3) does not follow directly from Lemma 2.3 below. In addition, as far as we know, it is the first time to give the L^∞ decay estimate $\partial_t u(t)$ in (1.2).

(3) The estimates (1.2), (1.3) improve greatly results in [20,23]. Indeed, the time behavior of $\|\nabla^\ell u(t)\|_{L^r(\Omega)}$ is derived only for $1 \le r \le 2$, $\ell = 1, 2$ in [23]; and $1 \le r \le \infty$, $\ell = 1$; $1 \le r < \infty$, $\ell = 2$ in [20] respectively. In particular, compared to decay results in [20], the decay rates of $\|\nabla^\ell u(t)\|_{L^r(\Omega)}$ ($\ell = 1, 2$) in Theorem 1.1 become faster for $2 < r < 4$. In addition, the estimates $\|\nabla u(t)\|_{L^\infty(\Omega)}$, $\|\nabla^2 u(t)\|_{L^r(\Omega)}$ with $4 \le r \le \infty$ are not considered in [20,23]. Exactly speaking, the following asymptotic behavior are established in [20,23]:

$$\|\nabla u(t)\|_{L^r(\Omega)} \le \begin{cases} Ct^{-\frac{1}{2}-(1-\frac{1}{r})} & \text{if } 1 \le r \le 2, \\ Ct^{-1} & \text{if } 2 < r < \infty, \end{cases}$$

and

$$\|\nabla^2 u(t)\|_{L^r(\Omega)} \le Ct^{-1} \ \text{ if } \ 1 < r < 4.$$

(4) It needs to point out that $u(t) \in L^1_w(\Omega) \backslash L^1(\Omega)$ for each fixed $t > 0$. In fact, recalling the representation of the matrix function $V(x,t)$ in Theorem 1.1, it is sufficient to prove $V(x,t) \in L^1_w(\Omega) \backslash L^1(\Omega)$ for every $t > 0$, where the norm of weak space $L^1_w(\Omega)$ is defined by

$$\|w\|_{L^1_w(\Omega)} = \sup_{\lambda > 0} \big(\lambda \, meas\{x \in \Omega : \ |w(x)| > \lambda\}\big).$$

Let $(x,t) \in \mathbb{R}^2 \times (0,+\infty)$. Then

$$\begin{aligned}
|V(x,t)| &\le |E_t(x)| + \Big| \int_0^\infty \nabla^2 E_{t+\tau}(x) d\tau \Big| \\
&\le C\Big(t^{-1} e^{-\frac{|x|^2}{4t}} + \int_t^\infty \tau^{-2} e^{-\frac{|x|^2}{8\tau}} d\tau \Big) \\
&\le C|x|^{-2} \Big(\frac{|x|^2}{4t} e^{-\frac{|x|^2}{4t}} + \int_0^{\frac{|x|^2}{t}} e^{-\frac{s}{8}} ds \Big) \\
&\le C|x|^{-2}. \qquad (1.5)
\end{aligned}$$

Set $f(x) = |x|^{-2}$. Then

$$\|f\|_{L^1_w(\Omega)} = \sup_{\lambda>0} \big(\lambda\, meas\{x \in \Omega : |x|^{-2} > \lambda\} \big) \le \pi,$$

which, together with (1.5) implies $V(x,t) \in L^1_w(\Omega)$ for each fixed $t > 0$.

On the other hand, take $R > 0$ large enough, such that $\mathbb{R}^2 \backslash \overline{\Omega} \subset B_R(0)$. It follows from (1.5) that for each $t > 0$

$$\begin{aligned}
\|V(\cdot,t)\|_{L^1(\mathbb{R}^2\backslash\overline{\Omega})} &\le \|V(\cdot,t)\|_{L^1(B_R(0))} \\
&\le C\Big(t^{-1} \int_{\mathbb{R}^2} e^{-\frac{|x|^2}{4t}} dx + \int_t^\infty \tau^{-2} \int_{B_R(0)} e^{-\frac{|x|^2}{8\tau}} dx d\tau \Big) \\
&\le C\Big(1 + \int_t^\infty \tau^{-1} \int_0^{\frac{R}{\sqrt{\tau}}} e^{-\frac{s^2}{8}} s ds d\tau \Big) \\
&\le C\Big(1 + R \int_t^\infty \tau^{-\frac{3}{2}} d\tau \int_0^\infty e^{-\frac{s^2}{8}} ds \Big) \\
&\le C(1 + Rt^{-\frac{1}{2}}),
\end{aligned}$$

which shows for each $t > 0$

$$V(x,t) \in L^1(\mathbb{R}^2\backslash\overline{\Omega}). \qquad (1.6)$$

Define the Fourier transform in $\mathbb{R}^2$ as follows:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-i\xi\cdot x} f(x) dx.$$

We find via the Fourier transform,

$$\widehat{V}(\xi,t) = \widehat{E_t}(\xi)Id + \int_0^\infty \widehat{\nabla^2 E_{t+s}}(\xi)ds$$
$$= e^{-t|\xi|^2} Id - \xi\otimes\xi \int_0^\infty e^{-(t+s)|\xi|^2} ds$$
$$= e^{-t|\xi|^2}\Big(Id - \frac{\xi\otimes\xi}{|\xi|^2}\Big), \quad t>0,$$

where Id denotes the 2×2 unit matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, and $\xi = (\xi_1,\xi_2)$, $\xi\otimes\xi$ means 2×2 matrix $\begin{pmatrix}\xi_1\xi_1 & \xi_1\xi_2\\ \xi_1\xi_2 & \xi_2\xi_2\end{pmatrix}$.

Set $g_{jk}(\xi) = \frac{\xi_j\xi_k}{|\xi|^2}$, $\xi = (\xi_1,\xi_2)$, $j,k = 1,2$. Then

$$\lim_{\substack{\xi\longrightarrow 0\\ \xi_1=\xi_2\neq 0}} g_{12}(\xi) = 1 \quad\text{and}\quad \lim_{\substack{\xi\longrightarrow 0\\ \xi_1=-\xi_2\neq 0}} g_{12}(\xi) = -1.$$

This shows that $g_{12}(\xi)$ is not continuous at $\xi = 0$, and so $\widehat{V}(\xi,t)$ is also not continuous at $\xi = 0$ for every $t > 0$. The arguments tell us that $V(x,t) \notin L^1(\mathbb{R}^2)$ for each $t > 0$. Whence from (1.6), $V(x,t)\notin L^1(\Omega)$ for any $t>0$. Therefore, it follows from (1.4) with $(r,\alpha) = (1,0)$ that $u(x,t)\notin L^1(\Omega)$ for large time $t>0$. Meanwhile, it reveals that the total net force $\int_0^\infty \mathcal{F}(\tau)d\tau$ plays a crucial role in determining the L^1-summability in the case of $(r,\alpha) = (1,0)$ in (1.4).

The following result is intended as an attempt at studying large time asymptotic profiles of the Navier–Stokes flows, which, together with the estimate (1.4) ($\alpha = 0$), reveals the reason that which term decides the decay rates of Navier–Stokes flows.

Theorem 1.2. *Assume $a \in L^1(\Omega)\bigcap L^2_\sigma(\Omega)\bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1 < q \le 2$. Let u be the solution of (1.1), given in Theorem 1.1. If $(1+|x|)a \in L^1(\Omega)$, then for $1\le r\le 2$,*

$$\lim_{t\longrightarrow\infty} \frac{t^{\frac{1}{2}+(1-\frac{1}{r})}}{\log_e(1+t)}\Big\| u(t) - V(x,t)\cdot\int_0^t \mathcal{F}(\tau)d\tau$$
$$+\nabla V(x,t)\cdot\int_0^t\int_\Omega (u\otimes u)(y,\tau)dyd\tau\Big\|_{L^r(\Omega)} = 0.$$

Remark. If we additionally assume $\int_0^\infty \|u(s)\|^2_{L^2(\Omega)}ds < \infty$, then for $1\le r\le\infty$ and $t>0$

$$\begin{aligned}&\|\nabla V(x,t)\cdot\int_0^t\int_\Omega(u\otimes u)(y,\tau)dyd\tau\|_{L^r(\Omega)}\\&\le\|\nabla V(\cdot,t)\|_{L^r(\Omega)}\int_0^t\int_\Omega|(u\otimes u)(y,\tau)|dyd\tau\\&\le Ct^{-\frac{1}{2}-(1-\frac{1}{r})}\int_0^\infty\|u(s)\|^2_{L^2(\Omega)}ds\\&\le Ct^{-\frac{1}{2}-(1-\frac{1}{r})},\end{aligned}$$

which shows for $1\le r\le 2$

$$\lim_{t\longrightarrow\infty}\frac{t^{\frac{1}{2}+(1-\frac{1}{r})}}{\log_e(1+t)}\Big\|\nabla V(x,t)\cdot\int_0^t\int_\Omega(u\otimes u)(y,\tau)dyd\tau\Big\|_{L^r(\Omega)}=0.$$

Whence, the expression in Theorem 1.2: $\nabla V(x,t)\cdot\int_0^t\int_\Omega(u\otimes u)(y,\tau)dyd\tau$ can be removed under the additional assumption: $\int_0^\infty\|u(s)\|^2_{L^2(\Omega)}ds<\infty$.

The third main result in this article devotes to large time behavior of higher-order norms of Navier–Stokes flows of (1.1), and this is a challenging problem for a long time. Schonbek and Wiegner [37] studied this topic in $\mathbb{R}^n$ in terms of its energy decay, where the proof of the results relies upon some energy estimates for higher-order spatial derivatives, the known L^∞ decay estimate and the Fourier splitting method due to M. Schonbek [35]. However, the situation changes in the case of the exterior domain Ω with compact boundary $\partial\Omega$. The main difficulties come from the lack of the weighted estimates with respect to velocity and pressure because of the appearance of the domain boundary. These mentioned arguments on the (weighted) decay results in the whole space cannot be applied to our present case because the domain boundary $\partial\Omega$ is nonempty, and $\partial P\neq P\partial$ in the exterior domain Ω, which results in many difficulties (for example, strong singularity appears) in dealing with the asymptotic behavior of higher-order derivatives for the solution of (1.1).

Theorem 1.3. *Let the exterior domain Ω be of class C^k with integer $k\ge 1$, and $a\in L^1(\Omega)\bigcap L^2_\sigma(\Omega)\bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1<q\le 2$. Let u be the strong solution of (1.1), which is given in Theorem 1.1. Then there exists $t_k>0$ such that for $t\ge t_k$*

$$\|\nabla^{2+k}u(t)\|_{L^r(\Omega)}\le Ct^{-\frac{3}{2}},\ \ 1\le r\le\infty.\tag{1.7}$$

Remark. (1) To the best of our knowledge, it is the first time to give the decay results for the higher order spatial derivatives of the strong solution of (1.1). The main difficulty lies that the L^q-L^r estimates for higher-order derivatives of the Stokes flow are not known up to now, and we cannot directly estimate $\|\nabla^{2+k}u(t)\|_{L^r(\Omega)}$ with $k \geq 1$ in the integral equality on the strong solution u of (1.1). Fortunately such difficulty can be overcome by applying regularity estimates on steady Stokes system. In addition, it should be pointed out that decays in time of $\|\nabla^\ell u(t)\|_{L^r(\Omega)}$ ($\ell = 0, 1, 2$) have been established in [20] under the appropriate assumptions on initial data.

(2) In the estimate (3.67) below, we establish the following decay result for each integer $k \geq 1$ and large time t:

$$\|\nabla^{2+k}u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k}p(t)\|_{L^r(\Omega)} \leq \begin{cases} Ct^{-1-(1-\frac{1}{r})} & \text{if } 1 < r \leq 2, \\ Ct^{-\frac{3}{2}} & \text{if } 2 < r \leq \infty. \end{cases}$$

Compared to the decay rate $t^{-\frac{3}{2}}$ in (1.7), the above estimate $t^{-1-(1-\frac{1}{r})} = t^{-\frac{3}{2}+(\frac{1}{r}-\frac{1}{2})}$ for $1 < r < 2$ becomes slower for large time t. In other words, the decay rate $t^{-\frac{3}{2}}$ in (1.7) is better than that in (3.67). In addition, the large time behavior of higher-order norms of the associated pressure function p is also given in (3.67).

It seems reasonable to believe that (1.7) can be further improved, namely, whether the following estimate is true for every integer $k \geq 1$ and large time t

$$\|\nabla^{2+k}u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k}p(t)\|_{L^r(\Omega)} \leq Ct^{-\frac{2+k}{2}-(1-\frac{1}{r})}. \tag{1.8}$$

Up to now, this is still an open question. In fact, in the proof of (1.7), we make use of the regularity estimate (2.7) for the steady Stokes system, and find for integer $k \geq 1$, $1 < r < \infty$ and $t > 0$

$$\|\nabla^{2+k}u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k}p(t)\|_{L^r(\Omega)} \leq C(\|(u \cdot \nabla u)(t)\|_{W^{k,r}(\Omega)} + \|\partial_t u(t)\|_{W^{k,r}(\Omega)}). \tag{1.9}$$

The estimate in the right hand side of (1.9): $\|(u \cdot \nabla u)(t)\|_{W^{k,r}(\Omega)} + \|\partial_t u(t)\|_{W^{k,r}(\Omega)}$ contains one bad term, $\|(u \cdot \nabla u)(t)\|_{L^r(\Omega)} + \|\partial_t u(t)\|_{L^r(\Omega)}$, which makes the decay of $\|\nabla^{2+k}u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k}p(t)\|_{L^r(\Omega)}$ to become slower for large time t. Actually if we can replace the estimate $\|(u \cdot \nabla u)(t)\|_{W^{k,r}(\Omega)} + \|\partial_t u(t)\|_{W^{k,r}(\Omega)}$ by $\|\nabla^k(u \cdot \nabla u)(t)\|_{L^r(\Omega)} + \|\nabla^k \partial_t u(t)\|_{L^r(\Omega)}$ in (1.9), checking the proof of Theorem 1.3, we find that the desired estimate (1.8) is really valid. Unfortunately, at present we cannot find technical approaches and innovative ideas to improve the known regularity estimate (2.7). In other words, we are not sure whether (2.7) is true after $\|f\|_{W^{k,r}(\Omega)}$ being replaced by $\|\nabla^k f\|_{L^r(\Omega)}$.

Now we turn to the weighted decay cases for Navier–Stokes flows. Many mathematicians have paid attention to this kind of topic in recent years. To Cauchy problem, He and Xin [25], M. Schonbek and T. Schonbek [36] established the following estimate for the weak, strong solution u of 3D Navier–Stokes equations, respectively, for $t > 0$:

$$\||x|^\alpha u(t)\|_{L^2(\mathbb{R}^3)}^2 + \int_0^t \||x|^\alpha \nabla u(s)\|_{L^2(\mathbb{R}^3)}^2 ds \le C \tag{1.10}$$

with $\alpha = \frac{3}{2}$, under assumptions: $a \in L^1(\mathbb{R}^3) \bigcap L^2_\sigma(\mathbb{R}^3)$ and $|x|^{\frac{3}{2}} a \in L^2(\mathbb{R}^3)$. Bae and Jin [3] proved (1.10) for weak solutions with $1 < \alpha < \frac{5}{2}$, assuming $a \in L^2_\sigma(\mathbb{R}^3)$, $(1+|x|)a \in L^1(\mathbb{R}^3)$ and $|x|^\alpha a \in L^2(\mathbb{R}^3)$. Brandolese [14] found a local smooth solution $u \in C([0,T); Z_\alpha)$ with some $T > 0$, assuming $a \in Z_\alpha$ for $\frac{3}{2} < \alpha < \frac{9}{2}$ ($\alpha \ne \frac{5}{3}, \frac{7}{2}$). Here $v \in Z_\alpha$ means that $(1+|x|^2)^{\alpha-2} v$, $(1+|x|^2)^{\alpha-1}\nabla v$, $(1+|x|^2)^\alpha \nabla^2 v \in L^2(\mathbb{R}^3)$. Recently Bae and Jin [7] considered the decay rates of L^2-moments in a 3D exterior domain Ω, and they proved that there is a weak solution u of (1.1) such that if $a \in L^r(\Omega) \bigcap L^2_\sigma(\Omega)$ with some $1 < r < \frac{6}{5}$, $|x|a \in L^{\frac{6}{5}}(\Omega)$ and $|x|^2 a \in L^2(\Omega)$, then $\||x|u(t)\|_{L^2(\Omega)} \le C_\delta (1+t)^{\frac{5}{4} - \frac{3}{2r} + \delta}$ for small number $\delta > 0$.

The last result in this article devotes to weighted decay results of higher-order derivatives of the solution of (1.1). As far as we know, this is the first time to give the systematic study on the decay properties of two-dimensional Navier–Stokes flows in exterior domains.

Theorem 1.4. *Suppose $a \in L^1(\Omega) \bigcap L^2_\sigma(\Omega) \bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1 < q \le 2$. Let u be the solution of (1.1) given in Theorem 1.1. If $\int_\Omega |x|^\alpha |a(x)| dx < \infty$, $0 < \alpha < 1$, then for $1 \le r \le \infty$ and large time t*

$$\||x|^\alpha \nabla u(t)\|_{L^r(\Omega)} \le C t^{\frac{\alpha}{2} - \frac{1}{2} - (1-\frac{1}{r})}; \tag{1.11}$$

$$\||x|^\alpha \nabla^2 u(t)\|_{L^r(\Omega)} \le \begin{cases} C t^{\frac{\alpha}{2} - 1 - (1-\frac{1}{r})} & \text{if } \frac{\alpha}{2} - (1-\frac{1}{r}) \ge 0, \\ C(t^{-\frac{3}{2}} + t^{\frac{\alpha}{2} - 1 - (1-\frac{1}{r})}) & \text{if } \frac{\alpha}{2} - (1-\frac{1}{r}) < 0. \end{cases} \tag{1.12}$$

Further if the exterior domain Ω is of class C^k with integer $k \ge 1$, then

$$\||x|^\alpha \nabla^{2+k} u(t)\|_{L^r(\Omega)} \le C(t^{-\frac{3}{2}} + t^{\frac{\alpha}{2} - \frac{3}{2} - (1-\frac{1}{r})}). \tag{1.13}$$

Remark. (1) The weighted estimates of $\||x|^\alpha \nabla^\ell u(t)\|_{L^r(\Omega)}$ ($\ell = 1, 2$) are established in [20]. Compared to these known results in [20], the decay rates of (1.11), (1.12) with $2 < r < \infty$ become faster for large time t. Additionally, the estimates $\||x|^\alpha \nabla u(t)\|_{L^\infty(\Omega)}$, $\||x|^\alpha \nabla^2 u(t)\|_{L^r(\Omega)}$ with $4 \le r \le \infty$, and the case of (1.13) for $k \ge 1$ are also the first time to be considered in this article according to our knowledge.

(2) It is natural to ask whether the following weighted decay estimate is valid under the assumptions in Theorem 1.4: $\||x|^\alpha \nabla^{2+k} u(t)\|_{L^r(\Omega)} \le C t^{\frac{\alpha}{2} - \frac{2+k}{2} - (1-\frac{1}{r})}$. At present, it is impossible to answer this question, because we are not sure whether the weighted estimate (1.14) is true for the unique solution (w, π) of the stationary Stokes system: Let $1 < r < \infty$ and $\ell \ge 0$ be an integer. Then

$$\||x|^\alpha \nabla^{\ell+2} w\|_{L^r(\Omega)} + \||x|^\alpha \nabla^{\ell+1} \pi\|_{L^r(\Omega)} \le C(\||x|^\alpha \nabla^\ell f\|_{L^r(\Omega)} + \||x|^\alpha \nabla^{\ell+1} g\|_{L^r(\Omega)}). \tag{1.14}$$

Up to now, the validity of (1.14) is still unknown for the case of $\alpha = 0$. Even if we suppose that the estimate (1.14) holds, then applying (1.14) to problem (1.1), we find another extreme difficulty inevitably appears, that is, have to handle the crucial estimate $\||x|^\alpha \partial_t^m u(t)\|_{L^r(\Omega)}$ $(m = 1, 2, \cdots)$ for the (strong) solution u of (1.1), which, currently we really have no ideas to deal with.

The paper is organized as follows. In Section 2, we collect some basic and known properties regarding the Stokes semigroup and Navier–Stokes flows of (1.1), which are applied frequently in the subsequent sections. Some new inequalities are found, which plays an important role in dealing with the boundary integrals. Together with regularity theory on steady Stokes system, we establish large time behavior of second-order spatial derivatives of the solution of (1.1). In particular, L^1-summability of 2D Navier–Stokes flows is completely characterized, and the associated total net force exerted on the boundary is shown to be finite, see Theorem 1.1 and Lemma 2.5 for details. In fact, such a study is of independent interest. Additionally, the asymptotic profiles of the solution of (1.1) is given, see Theorem 1.2. In section 3, by means of elliptic estimates for the steady Stokes solution, we establish decays in time of higher-order norms of flows of (1.1), see Theorem 1.3. The aim of Section 4 is to study the weighted decays of higher-order derivatives of the solution of (1.1). To do this, splitting the exterior domain Ω into two parts, that is, $\Omega = \Omega_\delta \cup \big(\Omega \backslash \Omega_\delta\big)$, where $\Omega_\delta = \{x \in \Omega;\ dist(x, \partial\Omega) > \delta\}$. With the aid of the conclusions obtained in Theorem 1.3, we focus and work on the two sub-regions, respectively, then achieve the desired results, e.g. Theorem 1.4. In this article, symbol C means a generic positive constant whose value may change from line to line.

2. L^1-summability and asymptotic behavior of second-order norms

This section devotes to studying these topics: L^1-summability, asymptotic profiles and large time behavior of second-order norms for two-dimensional Navier–Stokes flows in the exterior domain. These questions have been attracting much attention, however only little progress is achieved in last decades, and so far, most of these questions are still not solved.

Note that the Stokes operator $-A$ generates a bounded analytic semigroup $\{e^{-tA}\}_{t\geq 0}$, and the function $e^{-tA}a$ solves the Stokes system, i.e., problem (1.1) with $u \cdot \nabla u$ deleted, with the corresponding initial value a. There are many important known results on the Stokes semigroup e^{-tA}, and regularity-decay estimates (like $L^q - L^r$ estimates) on exterior domains are extensively studied by many mathematicians in recent years. The following estimates reveal that some $L^q - L^r$ properties of the Stokes flow in the exterior domain Ω are different from those in the whole space $\mathbb{R}^2$.

Lemma 2.1. *(See [23].) Let $a \in L^q_\sigma(\Omega)$. Then for any $t > 0$*

$$\|e^{-tA}a\|_{L^r(\Omega)} \leq C_{q,r} t^{-(\frac{1}{q}-\frac{1}{r})}\|a\|_{L^q(\Omega)}, \quad \forall\, 1 \leq q < r \leq \infty \ \ or\ \ 1 < q \leq r < \infty;$$

$$\|\nabla e^{-tA}a\|_{L^r(\Omega)} \leq C_{q,r}t^{-\frac{1}{2}-(\frac{1}{q}-\frac{1}{r})}\|a\|_{L^q(\Omega)}, \quad \forall\, 1 \leq q < r \leq 2 \;\; or \;\; 1 < q \leq r \leq 2;$$
$$\|\nabla e^{-tA}a\|_{L^r(\Omega)} \leq C_{q,r}(1+t^{-\frac{1}{2}})t^{-(\frac{1}{q}-\frac{1}{r})}\|a\|_{L^q(\Omega)}, \quad \forall\, 1 < q \leq r, \;\; 2 < r < \infty.$$

Lemma 2.2. *(See [20,23].) Let $a \in L^2_\sigma(\Omega)$. There exists a unique smooth solution u of (1.1) on $\Omega \times (0,\infty)$, which satisfies*

$$\|u(t)\|^2_{L^2(\Omega)} + 2\int_s^t \|\nabla u(\tau)\|^2_{L^2(\Omega)}d\tau = \|u(s)\|^2_{L^2(\Omega)} \quad \text{for all } 0 \leq s \leq t.$$

Let p and p_0 denote, respectively the pressures associated with u and $u_0 = e^{-tA}a$. Then $w = u - u_0$, $\pi = p - p_0$ satisfy for all $0 < T < \infty$

$$\int_0^T \big(\|\partial_t w(t)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Omega)} + \|\nabla^2 w(t)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Omega)} + \|w(t)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Omega)} + \|\nabla\pi(t)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Omega)}\big)dt \leq C_T\|a\|^2_{L^2(\Omega)},$$

where C_T is independent of a and w.

Moreover, if $a \in W_0^{2-\frac{2}{r},r}(\Omega)$ for $1 < r \leq 2$, then (u_0,p_0) satisfies for each $0 < T < \infty$

$$\int_0^T \big(\|\partial_t u_0(t)\|^r_{L^r(\Omega)} + \|\nabla^2 u_0(t)\|^r_{L^r(\Omega)} + \|u_0(t)\|^r_{L^r(\Omega)} + \|\nabla p_0(t)\|^r_{L^r(\Omega)}\big)dt$$
$$\leq C_T\|a\|^r_{W^{2-\frac{2}{r},r}(\Omega)},$$

where C_T is independent of a and u_0.

Furthermore, if $a \in L^1(\Omega)$, then the solution (u,p) of (1.1) satisfies for all $t > 1$

$$\|u(t)\|_{L^q(\Omega)} \leq Ct^{-(1-\frac{1}{q})} \quad \text{with} \quad 1 < q \leq \infty,$$
$$\|\nabla u(t)\|_{L^q(\Omega)} \leq \begin{cases} Ct^{-\frac{1}{2}}\log t & \text{if } q = 1, \\ Ct^{-\frac{1}{2}-(1-\frac{1}{q})} & \text{if } 1 < q \leq 2, \\ Ct^{-1} & \text{if } 2 < q < \infty, \end{cases}$$
$$\|\nabla^2 u(t)\|_{L^q(\Omega)} \leq \begin{cases} Ct^{-1}\log_e(1+t) & \text{if } q = 1, \\ Ct^{-1} & \text{if } 1 < q < 4, \end{cases}$$
$$\|\nabla^2 u(t)\|_{L^q(\Omega)} + \|\nabla p(t)\|_{L^q(\Omega)} \leq Ct^{-1} \quad \text{for } 1 < q \leq 2,$$

and

$$\|Au(t)\|_{L^q(\Omega)} \leq Ct^{-1-(1-\frac{1}{q})} \quad \text{for } 1 < q < 4.$$

In the above estimate $\|Au(t)\|_{L^q(\Omega)}$, the parameter q is restricted to be $1 < q < 4$ for some technical reasons, such range is enlarged to be $1 < q < \infty$ in the following Lemma, which will be applied to establish the decay estimate of $\|\nabla^2 u(t)\|_{L^q(\Omega)}$ with $1 < q < \infty$.

Lemma 2.3. *Let $a \in L^1(\Omega) \bigcap L^2_\sigma(\Omega) \bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1 < q \leq 2$. Then the solution u of (1.1) given in Lemma 2.2 satisfies for any $t \geq 2$*

$$\|Au(t)\|_{L^r(\Omega)} \leq Ct^{-1-(1-\frac{1}{r})} \quad \text{for } 1 < r < \infty.$$

Proof. The case of $1 < r \leq 2$ in Lemma 2.3 has been verified in [23], and the case of $2 < r < 4$ is proved in [20]. Here we give the proof for all $1 < r < \infty$. It is shown in [20] that for $0 < \alpha < 1$ and $0 < \delta < 1 - \alpha$, there holds for any $t \geq 2$, $h \geq 0$

$$\|A^\alpha u(t+h) - A^\alpha u(t)\|_{L^2(\Omega)} \leq C\big(h^\delta t^{-\alpha-\delta-\frac{1}{2}} + h^{1-\alpha} t^{-2}\big). \tag{2.1}$$

Note that for any $t > 0$, the solution u of (1.1) with initial data $a \in L^2_\sigma(\mathbb{R}^n_+)$ can be expressed as follows:

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-s)A} Pu(s) \cdot \nabla u(s) ds,$$

and then

$$\begin{aligned} Au(t) &= Ae^{-\frac{3t}{4}A}u(\frac{t}{4}) - (I - e^{-\frac{t}{2}A})P(u \cdot \nabla u)(t) \\ &\quad - \int_{\frac{t}{4}}^{\frac{t}{2}} Ae^{-(t-s)A}P(u \cdot \nabla u)(s)ds \\ &\quad - \int_{\frac{t}{2}}^{t} Ae^{-(t-s)A}\big(P(u \cdot \nabla u)(s) - P(u \cdot \nabla u)(t)\big)ds \\ &= K_1(t) + K_2(t) + K_3(t) + K_4(t). \end{aligned} \tag{2.2}$$

The following estimates are obtained in [20] for any $t \geq 2$,

$$\|K_1(t)\|_{L^r(\Omega)} \leq Ct^{-2+\frac{1}{r}}; \quad \|K_2(t)\|_{L^r(\Omega)} \leq Ct^{-2}; \quad \|K_3(t)\|_{L^r(\Omega)} \leq Ct^{-\frac{5}{2}+\frac{1}{r}}. \tag{2.3}$$

Let $0 < \beta_1 < \frac{1}{2} < \beta_2 < 1$. Then it holds for any $w \in D(A)$

$$\|w\|_{L^\infty(\Omega)} \leq \|\int_0^L e^{-\frac{t}{2}A}A^{\beta_1}e^{-\frac{t}{2}A}A^{1-\beta_1}wdt\|_{L^\infty(\Omega)}$$

$$+\|\int_L^\infty e^{-\frac{t}{2}A}A^{\beta_2}e^{-\frac{t}{2}A}A^{1-\beta_2}wdt\|_{L^\infty(\Omega)}$$

$$\le C\int_0^L t^{-\frac{1}{2}}\|A^{\beta_1}e^{-\frac{t}{2}A}A^{1-\beta_1}w\|_{L^2(\Omega)}dt$$

$$+C\int_L^\infty t^{-\frac{1}{2}}\|A^{\beta_2}e^{-\frac{t}{2}A}A^{1-\beta_2}w\|_{L^2(\Omega)}dt$$

$$\le C\int_0^L t^{-\frac{1}{2}-\beta_1}\|A^{1-\beta_1}w\|_{L^2(\Omega)}dt$$

$$+C\int_L^\infty t^{-\frac{1}{2}-\beta_2}\|A^{1-\beta_2}w\|_{L^2(\Omega)}dt$$

$$\le C(L^{\frac{1}{2}-\beta_1}\|A^{1-\beta_1}w\|_{L^2(\Omega)}+L^{\frac{1}{2}-\beta_2}\|A^{1-\beta_2}w\|_{L^2(\Omega)})$$

$$\le C\|A^{1-\beta_1}w\|_{L^2(\Omega)}^{\frac{2\beta_2-1}{2(\beta_2-\beta_1)}}\|A^{1-\beta_2}w\|_{L^2(\Omega)}^{\frac{1-2\beta_1}{2(\beta_2-\beta_1)}}$$

by choosing $L=\big(\frac{\|A^{1-\beta_2}w\|_{L^2(\Omega)}}{\|A^{1-\beta_1}w\|_{L^2(\Omega)}}\big)^{\frac{1}{\beta_2-\beta_1}}$. In particular,

$$\|w\|_{L^\infty(\Omega)}\le C\|A^{\frac{1}{4}}w\|_{L^2(\Omega)}^{\frac{1}{2}}\|A^{\frac{3}{4}}w\|_{L^2(\Omega)}^{\frac{1}{2}}. \tag{2.4}$$

Using (2.4), we have for any $t>0$

$$\|u(t)\|_{L^\infty(\Omega)}\le C\|A^{\frac{1}{4}}u(t)\|_{L^2(\Omega)}^{\frac{1}{2}}\|A^{\frac{3}{4}}u(t)\|_{L^2(\Omega)}^{\frac{1}{2}}.$$

Therefore it follows from (2.1) and Lemmata 2.1, 2.2 that for any $t\ge 2$

$$\|K_4(t)\|_{L^r(\Omega)}\le C\int_{\frac{t}{2}}^t(t-s)^{-\frac{3}{2}+\frac{1}{r}}\|u(s)\|_{L^\infty(\Omega)}\|\nabla u(s)-\nabla u(t)\|_{L^2(\Omega)}ds$$

$$+C\int_{\frac{t}{2}}^t(t-s)^{-\frac{3}{2}+\frac{1}{r}}\|u(s)-u(t)\|_{L^\infty(\Omega)}\|\nabla u(t)\|_{L^2(\Omega)}ds$$

$$\le C\int_{\frac{t}{2}}^t(t-s)^{-\frac{3}{2}+\frac{1}{r}}s^{-1}\|A^{\frac{1}{2}}(u(s)-u(t))\|_{L^2(\Omega)}ds$$

$$
\begin{aligned}
&+ Ct^{-1}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{2}+\frac{1}{r}}\|A^{\frac{1}{4}}(u(s)-u(t))\|_{L^2(\Omega)}^{\frac{1}{2}}\|A^{\frac{3}{4}}(u(s)-u(t))\|_{L^2(\Omega)}^{\frac{1}{2}}ds\\
&\le Ct^{-1}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{2}+\frac{1}{r}}\Big((t-s)^{\delta_1}s^{-1-\delta_1}+(t-s)^{\frac{1}{2}}s^{-2}\\
&\quad+\big((t-s)^{\delta_2}s^{-\frac{3}{4}-\delta_2}+(t-s)^{\frac{3}{4}}s^{-2}\big)^{\frac{1}{2}}\big((t-s)^{\delta_3}s^{-\frac{5}{4}-\delta_3}+(t-s)^{\frac{1}{4}}s^{-2}\big)^{\frac{1}{2}}\Big)ds\\
&\le Ct^{-1}\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{2}+\frac{1}{r}}\Big((t-s)^{\delta_1}s^{-1-\delta_1}+(t-s)^{\frac{1}{2}}s^{-2}\\
&\quad+(t-s)^{\frac{\delta_2+\delta_3}{2}}s^{-1-\frac{\delta_2+\delta_3}{2}}+(t-s)^{\frac{\delta_2}{2}+\frac{1}{8}}s^{-\frac{11}{8}-\frac{\delta_2}{2}}+(t-s)^{\frac{\delta_3}{2}+\frac{3}{8}}s^{-\frac{13}{8}-\frac{\delta_3}{2}}\Big)ds\\
&\quad\Big(\text{here }\ 0<\delta_1<\frac{1}{2},\quad 0<\delta_2<\frac{3}{4},\quad 0<\delta_3<\frac{1}{4}\ \Big)\\
&\le Ct^{-\frac{3}{2}-(1-\frac{1}{r})}
\end{aligned}
\tag{2.5}
$$

by taking $\frac{1}{2}-\frac{1}{r}<\delta_1<\frac{1}{2}$, $\frac{1}{2}-\frac{1}{r}<\frac{\delta_2+\delta_3}{2}$, $\frac{3}{8}-\frac{1}{r}<\frac{\delta_2}{2}$ $\frac{1}{8}-\frac{1}{r}<\frac{\delta_3}{2}$ with $1<r<\infty$.

From (2.2), (2.3) and (2.5), we conclude that for any $1<r<\infty$ and $t\ge 2$

$$\|Au(t)\|_{L^r(\Omega)}\le Ct^{-1-(1-\frac{1}{r})}. \qquad \square$$

Now we recall a classical existence and regularity result on the steady Stokes equations in a general domain Q (see p. 35, Proposition 2.3 in section 2, Chapter 1 in [41]):

Let Q be a open domain of $\mathbb{R}^n$, $n=2$ or 3, of class $C^{\max\{m+2,2\}}$, integer $m\ge -1$, and let $f\in W^{m,q}(Q)$, $g\in W^{m+1,q}(Q)$, $\phi\in W^{m+2-\frac{1}{q},q}(\partial Q)$, $1<q<\infty$, be given satisfying the compatibility condition

$$\int_Q g(x)dx=\int_{\partial Q}\phi\cdot\nu dS. \tag{2.6}$$

Then there exist functions w and π (π is unique up to a constant), which are solutions of the steady Stokes system:

$$
\begin{cases}
-\Delta w+\nabla\pi=f & \text{in }\ Q,\\
\nabla\cdot w=g & \text{in }\ Q,\\
w(x)=\phi & \text{on }\ \partial Q.
\end{cases}
$$

Moreover, (w,π) obeys $u\in W^{m+2,q}(Q)$, $\pi\in W^{m+1,q}(Q)$, and the following estimate

$$
\begin{aligned}
&\|w\|_{W^{m+2,q}(Q)}+\|\pi\|_{W^{m+1,q}(Q)}\\
&\le c(n,m,q,Q)(\|f\|_{W^{m,q}(Q)}+\|g\|_{W^{m+1,q}(Q)}+\|\phi\|_{W^{m+2-\frac{1}{q},q}(\partial Q)}).
\end{aligned}
\tag{2.7}
$$

In this article, the uniqueness of solutions is always assumed to be established for the steady Navier–Stokes system in 2D exterior domains.

To proceed, we need to construct some types of useful inequalities on the exterior domain $\Omega \subset \mathbb{R}^2$, which are used frequently in subsequent sections.

Lemma 2.4. *Let $f \in W^{1,q}(\Omega)$. Then*

$$\|f\|_{L^{\frac{2q}{2-q}}(\Omega)} \leq C\|\nabla f\|_{L^q(\Omega)} \quad \textit{for} \quad 1 < q < 2, \tag{2.8}$$

$$\|f\|_{L^q(\partial\Omega)} \leq \begin{cases} C\|\nabla f\|_{L^q(\Omega)} & \textit{for } 1 < q < 2, \\ C(\|f\|_{L^\infty(\Omega)} + \|\nabla f\|_{L^q(\Omega)}) & \textit{for } 1 < q < \infty, \end{cases} \tag{2.9}$$

where $C = C(q, \partial\Omega)$, and the function f does not always vanish on the boundary $\partial\Omega$.

In addition, it holds for all $g \in W^{1,q}(\Omega)$

$$\|g\|_{L^q(\partial\Omega)} \leq C\|\nabla g\|_{L^q(\Omega)}, \tag{2.10}$$

here the function g satisfies $\int_{\Omega \bigcap B_{2R}(0)} g(x)dx = 0$ with some large lumber R satisfying $\mathbb{R}^2 \backslash \Omega \subset B_R(0)$, and $C = C(q, R, \partial\Omega)$.

Proof. Set $\Omega_{\delta,R} = \{x \in \Omega;\ dist(x, \partial\Omega) > \delta\} \bigcap \{x \in \mathbb{R}^2;\ |x| < R\}$, where $R > 0$ is sufficiently large and $\delta > 0$ sufficiently small. Let $\varphi_{\delta,R} \in C_0^\infty(\Omega_{\frac{\delta}{2},2R})$, $\varphi_{\delta,R} \equiv 1$ on $\overline{\Omega_{\delta,R}}$, $0 \leq \varphi_{\delta,R} \leq 1$ on $\mathbb{R}^2$. Extending smoothly $f\varphi_{\delta,R}$ to be 0 from $\Omega_{\frac{\delta}{2},2R}$ to $\mathbb{R}^2$.

Recall the classical Sobolev inequality on the whole space $\mathbb{R}^2$: Let $1 < q < 2$, it holds for any $g \in W^{1,q}(\mathbb{R}^2)$

$$\|g\|_{L^{\frac{2q}{2-q}}(\mathbb{R}^2)} \leq C\|\nabla g\|_{L^q(\mathbb{R}^2)}.$$

Let $f \in W^{1,q}(\Omega)$, $1 < q < 2$. Then we have with the above choice of numbers $R > 0$, $\delta > 0$

$$\begin{aligned} \|f\|_{L^{\frac{2q}{2-q}}(\Omega_{\delta,R})} &\leq \|f\varphi_{\delta,R}\|_{L^{\frac{2q}{2-q}}(\Omega_{\frac{\delta}{2},2R})} \leq \|f\varphi_{\delta,R}\|_{L^{\frac{2q}{2-q}}(\mathbb{R}^2)} \\ &\leq C\|\nabla(f\varphi_{\delta,R})\|_{L^q(\mathbb{R}^2)} \\ &\leq C(\|\varphi_{\delta,R}\nabla f\|_{L^q(\mathbb{R}^2)} + \|f\nabla\varphi_{\delta,R}\|_{L^q(\mathbb{R}^2)}) \\ &\leq C(\|\nabla f\|_{L^q(\Omega)} + (R-\delta)^{-1}\|f\|_{L^q(\Omega)}), \end{aligned} \tag{2.11}$$

where the constant $C = C(q) > 0$ is independent of δ, R. Taking $(\delta, R) \longrightarrow (0, \infty)$ in (2.11), we get the desired inequality (2.8).

Recalling the choice of number $R > 0$ with the property: $\mathbb{R}^2 \backslash \Omega \subset B_R(0)$, by using the classical Sobolev and trace inequalities, together with (2.8), we have for $f \in W^{1,q}(\Omega)$

$$\begin{aligned}\|f\|_{L^q(\partial\Omega)} &\le \|f\|_{L^q(\partial(\Omega\bigcap B_{2R}(0)))}\\ &\le C(\|f\|_{L^q(\Omega\bigcap B_{2R}(0))}+\|\nabla f\|_{L^q(\Omega\bigcap B_{2R}(0))})\\ &\le C(\|f\|_{L^{\frac{2q}{2-q}}(\Omega)}+\|\nabla f\|_{L^q(\Omega)})\\ &\le \widetilde{C}\|\nabla f\|_{L^q(\Omega)}, \quad 1<q<2. \end{aligned}\tag{2.12}$$

Meanwhile, we also get

$$\begin{aligned}\|f\|_{L^q(\partial\Omega)} &\le \|f\|_{L^q(\partial(\Omega\bigcap B_{2R}(0)))}\\ &\le C(\|f\|_{L^q(\Omega\bigcap B_{2R}(0))}+\|\nabla f\|_{L^q(\Omega\bigcap B_{2R}(0))})\\ &\le C(\|f\|_{L^\infty(\Omega\bigcap B_{2R}(0))}+\|\nabla f\|_{L^q(\Omega)})\\ &\le C(\|f\|_{L^\infty(\Omega)}+\|\nabla f\|_{L^q(\Omega)}), \quad 1<q<\infty. \end{aligned}\tag{2.13}$$

Combining (2.12) and (2.13) yields (2.9).

Let $g\in W^{1,q}(\Omega)$ satisfying $\int_{\Omega\bigcap B_{2R}(0)} g(x)dx=0$, where $R>0$ is chosen in the proof of (2.8). Checking the proof of (2.13), and using Poincaré inequality, we find

$$\begin{aligned}\|g\|_{L^q(\partial\Omega)} &\le \|g\|_{L^q(\partial(\Omega\bigcap B_{2R}(0)))}\\ &\le C(\|g\|_{L^q(\Omega\bigcap B_{2R}(0))}+\|\nabla g\|_{L^q(\Omega\bigcap B_{2R}(0))})\\ &\le C\|\nabla g\|_{L^q(\Omega)}, \quad 1<q<\infty, \end{aligned}$$

which is (2.10). □

Lemma 2.5. *Suppose $a\in L^1(\Omega)\bigcap L^2_\sigma(\Omega)\bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1<q\le 2$. Then the solution (u,p) of (1.1), given in Lemma 2.2, satisfies for $t\ge T_0$ with some number $T_0>2$*

$$\int_0^\infty \|T[u,p](s)\|_{L^1(\partial\Omega)}ds<\infty;\quad \|\nabla u(t)\|_{L^r(\Omega)}\le Ct^{-\frac12-(1-\frac1r)},\ 1\le r\le\infty,$$

and

$$\|\nabla^2 u(t)\|_{L^r(\Omega)}\le\begin{cases} Ct^{-1-(1-\frac1r)} & \text{if } 1\le r<\infty,\\ Ct^{-\frac74} & \text{if } \ r=\infty.\end{cases}$$

Proof. Let (u,p) be the strong solution of problem (1.1), given in Lemma 2.2. Then by Lemmata 2.2, 2.3, we get for $1<r<\infty$ and $t>1$

$$\begin{aligned}\|\partial_t u(t)\|_{L^r(\Omega)} &\le \|Au(t)\|_{L^r(\Omega)}+\|P(u\cdot\nabla u)(t)\|_{L^r(\Omega)}\\ &\le \|Au(t)\|_{L^r(\Omega)}+\|u(t)\|_{L^\infty(\Omega)}\|\nabla u(t)\|_{L^r(\Omega)}\end{aligned}$$

$$
\begin{aligned}
&\le Ct^{-1-(1-\frac{1}{r})} + \begin{cases} Ct^{-\frac{3}{2}}\log t & \text{if } r=1,\\ Ct^{-\frac{3}{2}-(1-\frac{1}{r})} & \text{if } 1<r\le 2,\\ Ct^{-2} & \text{if } \ 2<r<\infty, \end{cases}\\
&\le Ct^{-1-(1-\frac{1}{r})}.
\end{aligned} \tag{2.14}
$$

It follows from problem (1.1) that the strong solution (u,p) satisfies for all $t>0$

$$
\begin{cases} -\Delta u + \nabla p = F(u) & \text{in } \ \Omega,\\ \nabla\cdot u = 0 & \text{in } \ \Omega,\\ u=0 & \text{on } \ \partial\Omega,\\ u\longrightarrow 0 & \text{as } \ |x|\to\infty, \end{cases} \tag{1.1$'$}
$$

where $F(u) = -\partial_t u - (u\cdot\nabla)u$.

Using (2.14) and Lemma 2.2, we get for $1<r<\infty$ and $t\ge 2$

$$
\|F(u(t))\|_{L^r(\Omega)} \le \|\partial_t u(t)\|_{L^r(\Omega)} + \|(u\cdot\nabla u)(t)\|_{L^r(\Omega)} \le Ct^{-1-(1-\frac{1}{r})}.
$$

Obviously the uniqueness of solutions and compatibility condition (2.6) are valid for problem (1.1′). Whence, applying the regularity estimate (2.7) to problem (1.1′), using (2.14) and Lemmata 2.2, 2.3, we get for $1<r<\infty$ and $t>1$

$$
\begin{aligned}
&\|\partial_t u(t)\|_{L^r(\Omega)} + \|\nabla^2 u(t)\|_{L^r(\Omega)} + \|\nabla p(t)\|_{L^r(\Omega)}\\
&\le C(\|\partial_t u(t)\|_{L^r(\Omega)} + \|(u\cdot\nabla u)(t)\|_{L^r(\Omega)})\\
&\le C(\|Au(t)\|_{L^r(\Omega)} + (1+\|P\|_{\mathcal{L}(L^r\to L^r_\sigma)})\|u(t)\|_{L^\infty(\Omega)}\|\nabla u(t)\|_{L^r(\Omega)})\\
&\le Ct^{-1-(1-\frac{1}{r})}.
\end{aligned} \tag{2.15}
$$

To proceed, we recall one useful inequality, which is about the L^∞ estimate in the exterior domain $\Omega\subset\mathbb{R}^2$ (see [21]): Let $2<s<\infty$. Then for any $f\in W^{1,s}(\Omega)$

$$
\|f\|_{L^\infty(\Omega)} \le C(s)\|f\|^{1-\frac{2}{s}}_{L^s(\Omega)}\|\nabla f\|^{\frac{2}{s}}_{L^s(\Omega)}. \tag{2.16}
$$

It is known that for $g\in W^{1,2}(\mathbb{R}^2)$ (see [41])

$$
\|g\|_{L^4(\mathbb{R}^2)} \le C\|g\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla g\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}. \tag{2.17}
$$

Let $f\in W^{1,2}(\Omega)$. Extending smoothly $f\varphi_{\delta,R}$ to be 0 from $\Omega_{\frac{\delta}{2},2R}$ to $\mathbb{R}^2$. Then together with (2.17), we have

$$
\begin{aligned}
\|f\|_{L^4(\Omega_{\delta,R})} &\le \|f\varphi_{\delta,R}\|_{L^4(\Omega_{\frac{\delta}{2},2R})} \le \|f\varphi_{\delta,R}\|_{L^4(\mathbb{R}^2)}\\
&\le C\|f\varphi_{\delta,R}\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}\|\nabla(f\varphi_{\delta,R})\|^{\frac{1}{2}}_{L^2(\mathbb{R}^2)}
\end{aligned}
$$

$$\le C\|f\|^{\frac{1}{2}}_{L^2(\Omega)}(\|\varphi_{\delta,R}\nabla f\|_{L^2(\mathbb{R}^2)} + \|f\nabla\varphi_{\delta,R}\|_{L^q(\mathbb{R}^2)})^{\frac{1}{2}}$$
$$\le C\|f\|^{\frac{1}{2}}_{L^2(\Omega)}(\|\nabla f\|_{L^2(\Omega)} + (R-\delta)^{-1}\|f\|_{L^2(\Omega)})^{\frac{1}{2}}, \tag{2.18}$$

where the cut-off function $\varphi_{\delta,R}$ is from the proof process in Lemma 2.4.

Taking $(\delta, R) \longrightarrow (0,\infty)$ in (2.18), we get for $f \in W^{1,2}(\Omega)$

$$\|f\|_{L^4(\Omega)} \le C\|f\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla f\|^{\frac{1}{2}}_{L^2(\Omega)}. \tag{2.19}$$

Combining (2.15), (2.16), (2.19) and Lemma 2.2, we have for $t > 1$

$$\begin{aligned}\|\nabla u(t)\|_{L^\infty(\Omega)} &\le C\|\nabla u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\|\nabla^2 u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\\ &\le C(\|\nabla u(t)\|_{L^2(\Omega)}\|\nabla^2 u(t)\|_{L^2(\Omega)})^{\frac{1}{4}}\|\nabla^2 u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\\ &\le Ct^{(-\frac{1}{2}-(1-\frac{1}{2})-1-(1-\frac{1}{2}))\times\frac{1}{4}+(-1-(1-\frac{1}{4}))\times\frac{1}{2}} \le Ct^{-\frac{3}{2}}.\end{aligned} \tag{2.20}$$

Using Lemmata 2.2, 2.4, we get

$$\begin{aligned}&\int_0^1 \|T[u,p](s)\|_{L^1(\partial\Omega)}ds\\ &\le C(\int_0^1 \|T[u_0,p_0](s)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\partial\Omega)}ds)^{\frac{3}{4}}\\ &\quad + C(\int_0^1 \|T[w,\pi](s)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\partial\Omega)}ds)^{\frac{3}{4}}\\ &\le C(\int_0^1 (\|\nabla^2 u_0(s)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Omega)} + \|\nabla p_0(s)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Omega)})ds)^{\frac{1}{q}}\\ &\quad + C(\int_0^1 (\|\nabla^2 w(s)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Omega)} + \|\nabla \pi(s)\|^{\frac{4}{3}}_{L^{\frac{4}{3}}(\Omega)})ds)^{\frac{3}{4}}\\ &\le C,\end{aligned}$$

where the functions u_0, p_0, w, π are from Lemma 2.2.

Using (2.15) with $r = \frac{4}{3}$, Hölder inequality, and Lemmata 2.2, 2.4, we have

$$\int_1^\infty \|T[u,p](s)\|_{L^1(\partial\Omega)}ds$$

$$\leq C \int_1^\infty \|T[u,p](s)\|_{L^{\frac{4}{3}}(\partial\Omega)} ds$$

$$\leq C \int_1^\infty (\|\nabla^2 u(s)\|_{L^{\frac{4}{3}}(\Omega)} + \|\nabla p(s)\|_{L^{\frac{4}{3}}(\Omega)}) ds$$

$$\leq C \int_1^\infty s^{-1-(1-\frac{3}{4})} ds \leq C.$$

Whence we get

$$\int_0^\infty \|T[u,p](s)\|_{L^1(\partial\Omega)} ds = \Big(\int_0^1 + \int_1^\infty \Big) \|T[u,p](s)\|_{L^1(\partial\Omega)} ds \leq C. \tag{2.21}$$

Note that the strong solution (u,p) of (1.1) can be written as follows for $t > 0$

$$\begin{aligned} u(x,t) = &\int_\Omega E_t(x-y)a(y)dy + \int_0^t \int_{\partial\Omega} V(x-y,t-\tau)(T[u,p]\cdot\nu)(y,\tau)dS_y d\tau \\ &- \int_0^t \int_\Omega \nabla_x V(x-y,t-\tau)\cdot(u\otimes u)(y,\tau)dyd\tau, \end{aligned} \tag{2.22}$$

where the definitions of $T[u,p]$, $V(x,t)$, $E_t(x)$ are given in Theorem 1.1. Moreover, for $t > 0$

$$\||\cdot|^\alpha E_t(\cdot)\|_{L^q(\mathbb{R}^2)} + \||\cdot|^\alpha V(\cdot,t)\|_{L^q(\mathbb{R}^2)} \leq Ct^{\frac{\alpha}{2}-(1-\frac{1}{q})} \tag{2.23}$$

provided $1 < q \leq \infty$, $\alpha > 0$ and $\frac{\alpha}{2} - (1-\frac{1}{q}) < 0$;

$$\||\cdot|^\alpha \nabla^k E_t(\cdot)\|_{L^q(\mathbb{R}^2)} + \||\cdot|^\alpha \nabla^k V(\cdot,t)\|_{L^q(\mathbb{R}^2)} \leq Ct^{\frac{\alpha}{2}-\frac{k}{2}-(1-\frac{1}{q})} \tag{2.24}$$

for all $1 \leq q \leq \infty$, $\alpha > 0$ and $\frac{\alpha}{2} - \frac{k}{2} - (1-\frac{1}{q}) < 0$, $k = 1, 2, \cdots$.

Using (2.15), (2.21), (2.22)–(2.24) and Lemma 2.4, we conclude for $t > 2$

$$\begin{aligned} \|\nabla u(t)\|_{L^1(\Omega)} \leq &\int_\Omega \|\nabla E_t(\cdot - y)\|_{L^1(\Omega)} |a(y)| dy \\ &+ \int_0^t \int_{\partial\Omega} \|\nabla V(\cdot - y, t-\tau)\|_{L^1(\Omega)} |T[u,p]|(y,\tau) dS_y d\tau \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_\Omega \|\nabla V(\cdot-y,t-\tau)\|_{L^1(\Omega)}|(u\cdot\nabla u)(y,\tau)|dyd\tau\\
&\le Ct^{-\frac{1}{2}}\|a\|_{L^1(\Omega)}+Ct^{-\frac{1}{2}}\int_0^{\frac{t}{2}}\|T[u,p](\tau)\|_{L^1(\partial\Omega)}d\tau\\
&+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}}(\|\nabla^2u(\tau)\|_{L^{\frac{4}{3}}(\Omega)}+\|\nabla p(\tau)\|_{L^{\frac{4}{3}}(\Omega)})d\tau\\
&+Ct^{-\frac{1}{2}}\int_0^{\frac{t}{2}}\|u(\tau)\|_{L^2(\Omega)}\|\nabla u(\tau)\|_{L^2(\Omega)}d\tau\\
&+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}}\|u(\tau)\|_{L^2(\Omega)}\|\nabla u(\tau)\|_{L^2(\Omega)}d\tau\\
&\le Ct^{-\frac{1}{2}}+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}}\tau^{-1-(1-\frac{3}{4})}d\tau+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}}\tau^{-\frac{3}{2}}d\tau\\
&\le \widetilde{C}t^{-\frac{1}{2}}. \qquad (2.25)
\end{aligned}
$$

Using (2.20), (2.25), we get for $1<r<\infty$ and $t>2$

$$
\begin{aligned}
\|\nabla u(t)\|_{L^r(\Omega)}&\le\|\nabla u(t)\|_{L^\infty(\Omega)}^{1-\frac{1}{r}}\|\nabla u(t)\|_{L^1(\Omega)}^{\frac{1}{r}}\le Ct^{-\frac{3}{2}(1-\frac{1}{r})-\frac{1}{2r}}\\
&\le Ct^{-\frac{1}{2}-(1-\frac{1}{r})}. \qquad (2.26)
\end{aligned}
$$

Let $\delta>2\sup_{y\in\partial\Omega}|y|$, and $\Omega_\delta=\{x\in\Omega\,|\,dist(x,\partial\Omega)>\delta\}$. Then from (2.15), (2.21), (2.22) and Lemmata 2.2, 2.4, one has for $t>2$

$$
\begin{aligned}
\|\nabla^2u(t)\|_{L^1(\Omega_\delta)}&\le\int_\Omega\|\nabla^2E_t(\cdot-y)\|_{L^1(\Omega)}|a(y)|dy\\
&+\int_0^t\int_{\partial\Omega}\|\nabla^2V(\cdot-y,t-\tau)\|_{L^1(\Omega_\delta)}|T[u,p]|(y,\tau)dS_yd\tau\\
&+\int_0^{\frac{t}{2}}\int_\Omega\|\nabla^2V(\cdot-y,t-\tau)\|_{L^1(\Omega)}|(u\cdot\nabla u)(y,\tau)|dyd\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_{\frac{t}{2}}^{t}\int_{\Omega} \|\nabla V(\cdot - y, t-\tau)\|_{L^1(\Omega)}\big(|u(y,\tau)||\nabla^2 u(y,\tau)| + |\nabla u(y,\tau)|^2\big)dyd\tau \\
&\le Ct^{-1}\|a\|_{L^1(\Omega)} + Ct^{-1}\int_0^{\frac{t}{2}} \|T[u,p](\tau)\|_{L^1(\partial\Omega)}d\tau \\
&+ C_\delta \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-1}(\|\nabla^2 u(\tau)\|_{L^{\frac{4}{3}}(\Omega)} + \|\nabla p(\tau)\|_{L^{\frac{4}{3}}(\Omega)})d\tau \\
&+ Ct^{-1}\int_0^{\frac{t}{2}} \|u(\tau)\|_{L^2(\Omega)}\|\nabla u(\tau)\|_{L^2(\Omega)}d\tau \\
&+ C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}}\big(\|u(\tau)\|_{L^2(\Omega)}\|\nabla^2 u(\tau)\|_{L^2(\Omega)} + \|\nabla u(\tau)\|^2_{L^2(\Omega)}\big)d\tau \\
&\le Ct^{-1} + C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{-1}\tau^{-1-(1-\frac{3}{4})}d\tau \\
&+ Ct^{-1}(\int_0^1 \|u(\tau)\|^2_{L^2(\Omega)}d\tau)^{\frac{1}{2}}(\int_0^1 \|\nabla u(\tau)\|^2_{L^2(\Omega)}d\tau)^{\frac{1}{2}} \\
&+ Ct^{-1}\int_1^{\frac{t}{2}} \tau^{-\frac{3}{2}}d\tau + C\int_{\frac{t}{2}}^{t} (t-\tau)^{-\frac{1}{2}}\tau^{-2}d\tau \\
&\le \widetilde{C}t^{-1}. \qquad (2.27)
\end{aligned}
$$

On the other hand, from (2.15), we derive for $t > 2$

$$
\|\nabla^2 u(t)\|_{L^1(\Omega\setminus\overline{\Omega_\delta})} \le C\|\nabla^2 u(t)\|_{L^2(\Omega)} \le Ct^{-1-(1-\frac{1}{2})} \le Ct^{-1}. \qquad (2.28)
$$

Combining (2.27) and (2.28), one gets for $t > 2$

$$
\|\nabla^2 u(t)\|_{L^1(\Omega)} \le Ct^{-1}. \qquad (2.29)
$$

Now we show the decay estimate of $\|\nabla^2 u(t)\|_{L^\infty(\Omega)}$.

Using (2.7), (2.15), (2.16) and Lemma 2.4, we conclude for large time t

$$
\begin{aligned}
\|\nabla^2 u(t)\|_{L^\infty(\Omega)} &\le C\|\nabla^2 u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\|\nabla^3 u(t)\|^{\frac{1}{2}}_{L^4(\Omega)} \\
&\le C\|\nabla^2 u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\|\partial_t u(t) + (u\cdot\nabla u)(t)\|^{\frac{1}{2}}_{W^{1,4}(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
&\le C\|\nabla^2 u(t)\|_{L^4(\Omega)}^{\frac{1}{2}} (\|\nabla^2\partial_t u(t)\|_{L^{\frac{4}{3}}(\Omega)} + \|u(t))\|_{L^\infty(\Omega)}\|\nabla^2 u(t))\|_{L^4(\Omega)} \\
&\quad + \|\nabla u(t))\|_{L^8(\Omega)}^2 + \|\partial_t u(t)\|_{L^4(\Omega)} + \|u(t)\|_{L^\infty(\Omega)}\|\nabla u(t)\|_{L^4(\Omega)})^{\frac{1}{2}} \\
&\le Ct^{(-1-(1-\frac{1}{4}))\times\frac{1}{2}} \\
&\quad \times (t^{-2-(1-\frac{3}{4})} + t^{-2-(1-\frac{1}{4})} + t^{-1-2(1-\frac{1}{8})} + t^{-1-(1-\frac{1}{4})})^{\frac{1}{2}} \\
&\le Ct^{-\frac{7}{4}},
\end{aligned} \tag{2.30}
$$

where the decay estimate in time is employed: $\|\nabla^2\partial_t u(t)\|_{L^{\frac{4}{3}}(\Omega)} \le Ct^{-2-(1-\frac{3}{4})}$, $\forall\, t >> 1$ (see Lemma 3.3 below).

From (2.15), (2.20), (2.21), (2.25), (2.26), (2.29) and (2.30), we complete the proof of Lemma 2.5. □

The next Lemma devotes to giving L^1-summability and weighted decay estimates for the $2D$ Navier–Stokes flows in exterior domains.

Lemma 2.6. *Suppose $a \in L^1(\Omega)\bigcap L^2_\sigma(\Omega)\bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1 < q \le 2$. Let (u,p) be the solution of (1.1), given in Lemma 2.2. Additionally assume $|x|^{1+\alpha}a \in L^1(\Omega)$, $|x|^\alpha a \in L^2(\Omega)$ with $0 \le \alpha \le 1$. Then for $1 \le r \le \infty$, $(\alpha, r) \ne (1,1)$ and large time t*

$$
\Big\||x|^\alpha\big(u(t) - V(\cdot,t)\cdot\int_0^t \mathcal{F}(\tau)d\tau\big)\Big\|_{L^r(\Omega)} \le C\big(t^{-1} + t^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}\log_e(1+t)\big).
$$

Proof. Firstly we prove that if $a \in L^1(\Omega)\bigcap L^2_\sigma(\Omega)\bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1 < q \le 2$, then

$$
\int_\Omega a(y)dy = 0. \tag{2.31}
$$

Indeed, set $\widetilde{a}(x) = \begin{cases} a(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^2\backslash\Omega. \end{cases}$ Then $\widetilde{a} \in L^1(\mathbb{R}^2)\bigcap L^2(\mathbb{R}^2)\bigcap W^{2-\frac{2}{q},q}(\mathbb{R}^2)$ with $1 < q \le 2$. Moreover, it follows that for any scalar function $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$
\langle\nabla\cdot\widetilde{a},\varphi\rangle_{\mathbb{R}^2} = -\int_{\mathbb{R}^2}\widetilde{a}\cdot\nabla\varphi dx = -\int_\Omega a\cdot\nabla\varphi dx = \int_\Omega \varphi\nabla\cdot a\, dx - \int_{\partial\Omega}(a\cdot\nu)\varphi dS_x = 0,
$$

which shows $\nabla\cdot\widetilde{a} = 0$ in $\mathbb{R}^2$ in the sense of distribution. Whence, we have (see [30])

$$
\int_{\mathbb{R}^2}\widetilde{a}(y)dy = 0, \text{ and then } \int_\Omega a(y)dy = \int_{\mathbb{R}^2}\widetilde{a}(y)dy = 0,
$$

which is (2.31).

Note that $|x|^{1+\alpha}a \in L^1(\Omega)$, $|x|^\alpha a \in L^2(\Omega)$ with $0 \le \alpha \le 1$. Using (2.31), we find for $t \ge 1$

$$\begin{aligned}
&\big\||x|^\alpha \int_\Omega E_t(x-y)a(y)dy\big\|_{L^r(\Omega)} \\
&= \big\||x|^\alpha \int_\Omega (E_t(x-y)-E_t(x))a(y)dy\big\|_{L^r(\Omega)} \\
&= \big\||x|^\alpha \int_\Omega \int_0^1 \nabla E_t(x-sy)\cdot(-y)a(y)dsdy\big\|_{L^r(\Omega)} \\
&\le C\||\cdot|^\alpha \nabla E_t(\cdot)\|_{L^r(\mathbb{R}^2)} \int_\Omega |y||a(y)|dy \\
&\quad + C\|\nabla E_t\|_{L^r(\mathbb{R}^2)} \int_\Omega |y|^{1+\alpha}|a(y)|dy \\
&\le Ct^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}, \quad 1 \le r \le \infty;
\end{aligned} \tag{2.32}$$

and

$$\begin{aligned}
&\big\||x|^\alpha \int_0^t \int_\Omega \nabla_x V(x-y,t-\tau)\cdot(u\otimes u)(y,\tau)dyd\tau\big\|_{L^r(\Omega)} \\
&\le C\int_0^{\frac{t}{2}} \||\cdot|^\alpha \nabla V(\cdot,t-\tau)\|_{L^r(\mathbb{R}^2)}\|u(\tau)\|^2_{L^2(\Omega)}d\tau \\
&\quad + C\int_0^{\frac{t}{2}} \|\nabla V(\cdot,t-\tau)\|_{L^r(\mathbb{R}^2)}\|u(\tau)\|_{L^2(\Omega)}\||\cdot|^\alpha u(\cdot,\tau)\|_{L^2(\Omega)}d\tau \\
&\quad + C\int_{\frac{t}{2}}^t \||\cdot|^\alpha \nabla V(\cdot,t-\tau)\|_{L^1(\mathbb{R}^2)}\|u(\tau)\|^2_{L^{2r}(\Omega)}d\tau \\
&\quad + C\int_{\frac{t}{2}}^t \|\nabla V(\cdot,t-\tau)\|_{L^1(\mathbb{R}^2)}\|u(\tau)\|_{L^{2r}(\Omega)}\||\cdot|^\alpha u(\cdot,\tau)\|_{L^{2r}(\Omega)}d\tau \\
&\le Ct^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}\int_0^{\frac{t}{2}}(1+\tau)^{-1}d\tau + Ct^{-\frac{1}{2}-(1-\frac{1}{r})}\int_0^{\frac{t}{2}}(1+\tau)^{\frac{\alpha}{2}-1}d\tau
\end{aligned}$$

$$+ C \int_{\frac{t}{2}}^{t} \big((t-\tau)^{-\frac{1-\alpha}{2}}\tau^{-1-(1-\frac{1}{r})} + (t-\tau)^{-\frac{1}{2}}\tau^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}\big)d\tau$$

$$\le Ct^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}\log_e(1+t), \quad 1 \le r \le \infty, \quad (\alpha, r) \neq (1,1). \tag{2.33}$$

Observe that for $0 \le \alpha \le 1$ and $t > 0$

$$\big||x|^\alpha(\int_0^t \int_{\partial\Omega} V(x-y, t-\tau)T[u,p](y,\tau)\cdot \nu dS_y d\tau - V(x,t)\int_0^t \mathcal{F}(\tau)d\tau)\big|$$

$$\le C \int_0^t \int_{\partial\Omega} \int_0^1 |x-y\theta|^\alpha |\nabla_x V(x-y\theta, t-\tau)||y||T[u,p](y,\tau)|d\theta dS_y d\tau$$

$$+ C \int_0^t \int_{\partial\Omega} \int_0^1 |\nabla_x V(x-y\theta, t-\tau)||y|^{1+\alpha}\theta^\alpha |T[u,p](y,\tau)|d\theta dS_y d\tau$$

$$+ C \int_0^t \int_0^1 \tau|x|^\alpha|\partial_t V(x, t-\theta\tau)| \int_{\partial\Omega} |T[u,p](y,\tau)|dS_y d\theta d\tau$$

$$= J_1(x,t) + J_2(x,t) + J_3(x,t). \tag{2.34}$$

Next we estimate each term in (2.34). Using Lemmata 2.4, 2.5, one has for $0 \le \alpha \le 1$ and large time t

$$\|J_1(t)\|_{L^r(\Omega_\delta)} + \|J_2(t)\|_{L^r(\Omega_\delta)}$$

$$\le C \int_0^t \big(\||\cdot|^\alpha \nabla V(\cdot, t-\tau)\|_{L^r(\Omega_\delta)} + \|\nabla V(\cdot, t-\tau)\|_{L^r(\Omega_\delta)}\big)$$

$$\times \int_{\partial\Omega} |T[u,p](y,\tau)|dS_y d\tau$$

$$\le C(t^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})} + t^{-\frac{1}{2}-(1-\frac{1}{r})}) \int_0^{\frac{t}{2}} \int_{\partial\Omega} |T[u,p](y,\tau)|dS_y d\tau$$

$$+ C \int_{\frac{t}{2}}^t ((1+t-\tau)^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})} + (1+t-\tau)^{-\frac{1}{2}-(1-\frac{1}{r})})$$

$$\times (\|\nabla u(\tau)\|_{L^\infty(\Omega)} + \|\nabla^2 u(\tau)\|_{L^2(\Omega)} + \|\nabla p(\tau)\|_{L^2(\Omega)})$$

$$\le C(t^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})} + t^{-\frac{1}{2}-(1-\frac{1}{r})}) \int_0^\infty \|T[u,p](\tau)\|_{L^1(\partial\Omega)} d\tau$$

$$+ C\int_{\frac{t}{2}}^{t}((1+t-\tau)^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})} + (1+t-\tau)^{-\frac{1}{2}-(1-\frac{1}{r})})\tau^{-1-(1-\frac{1}{2})}d\tau$$

$$\leq Ct^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})} + Ct^{-\frac{3}{2}}\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}d\tau$$

$$\leq Ct^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}, \quad 1 \leq r \leq \infty, \quad (\alpha, r) \neq (1,1). \tag{2.35}$$

In the proof of (2.35), we made use of the estimates:

$$\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}d\tau \leq \begin{cases} Ct^{-\frac{1-\alpha}{2}+\frac{1}{r}} & \text{if } -\frac{1-\alpha}{2}+\frac{1}{r} > 0, \\ C\log_e(1+t) & \text{if } -\frac{1-\alpha}{2}+\frac{1}{r} = 0, \\ C & \text{if } -\frac{1-\alpha}{2}+\frac{1}{r} < 0, \end{cases}$$

and

$$t^{-\frac{3}{2}}\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}d\tau \leq Ct^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}, \quad \forall t > 1.$$

A direct calculation shows that for $1 \leq r \leq \infty$, $0 < \beta < 2 + 2(1-\frac{1}{r})$ and $t > 0$

$$\||\cdot|^{\beta}\partial_t V(\cdot,t)\|_{L^r(\mathbb{R}^2)} \leq \||\cdot|^{\beta}\partial_t E_t(\cdot)\|_{L^r(\mathbb{R}^2)} + \int_0^{\infty}\||\cdot|^{\beta}\partial_t\nabla^2 E_{t+\tau}(\cdot)\|_{L^r(\mathbb{R}^2)}d\tau$$

$$\leq Ct^{\frac{\beta}{2}-1-(1-\frac{1}{r})} + \int_0^{\infty}(t+\tau)^{\frac{\beta}{2}-2-(1-\frac{1}{r})}d\tau$$

$$\leq Ct^{\frac{\beta}{2}-1-(1-\frac{1}{r})}. \tag{2.36}$$

Using Lemmata 2.4, 2.5 and (2.36), we have for $0 \leq \alpha \leq 1$ and $t > 2N$ (N is a large number)

$$\|J_3(t)\|_{L^r(\Omega_\delta)} \leq C_\alpha \int_0^N\int_0^1 \tau\||\cdot|^{\alpha}\partial_t V(\cdot,t-\theta\tau)\|_{L^r(\mathbb{R}^2)}\int_{\partial\Omega}|T[u,p](y,\tau)|dS_y d\theta d\tau$$

$$+ C\Big(\int_N^{\frac{t}{2}} + \int_{\frac{t}{2}}^{t}\Big)\int_0^1 \tau\||\cdot|^{\alpha}\partial_t V(\cdot,t-\theta\tau)\|_{L^r(\Omega_\delta)}$$

$$\times (\|\nabla u(\tau)\|_{L^\infty(\Omega)} + \|\nabla^2 u(\tau)\|_{L^2(\Omega)} + \|\nabla p(\tau)\|_{L^2(\Omega)})d\theta d\tau$$

$$\leq C\int_0^N \tau(t-\tau)^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}\int_{\partial\Omega}|T[u,p](y,\tau)|dS_y d\tau$$

$$+C\int_N^{\frac{t}{2}} \tau(t-\tau)^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}\tau^{-1-(1-\frac{1}{2})}d\tau$$

$$+C\int_{\frac{t}{2}}^t\int_0^1 \tau(1+t-\theta\tau)^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}\tau^{-1-(1-\frac{1}{2})}d\theta d\tau$$

$$\leq Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}\int_0^N\int_{\partial\Omega}|T[u,p](y,\tau)|dS_y d\tau$$

$$+Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}\int_N^{\frac{t}{2}}\tau^{-\frac{1}{2}}d\tau+Ct^{-\frac{3}{2}}\int_{\frac{t}{2}}^t\int_0^1(1+t-\theta\tau)^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}\tau d\theta d\tau$$

$$\leq Ct^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})},\quad 1\leq r\leq\infty. \tag{2.37}$$

In the proof of (2.37), we made use of the estimates for $1\leq r\leq\infty$ and $t>2N$:

$$t^{-\frac{3}{2}}\int_{\frac{t}{2}}^t\int_0^1(1+t-\theta\tau)^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}\tau d\theta d\tau\leq Ct^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}. \tag{2.38}$$

Indeed, for $1\leq r\leq\infty$ and $t>2N$,

$$\int_{\frac{t}{2}}^t\int_0^1(1+t-\theta\tau)^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}\tau d\theta d\tau$$

$$=\int_{\frac{t}{2}}^t\int_0^\tau(1+t-\lambda)^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}d\lambda d\tau$$

$$\leq\begin{cases} Ct^{\frac{\alpha}{2}+1-(1-\frac{1}{r})} & \text{if } \frac{\alpha}{2}-(1-\frac{1}{r})>0,\\ C\int_{\frac{t}{2}}^t(\log_e(1+t)-\log_e(1+t-\tau))d\tau & \text{if } \frac{\alpha}{2}-(1-\frac{1}{r})=0,\\ C\int_{\frac{t}{2}}^t(1+t-\tau)^{\frac{\alpha}{2}-(1-\frac{1}{r})}d\tau & \text{if } \frac{\alpha}{2}-(1-\frac{1}{r})<0.\end{cases} \tag{2.39}$$

If $\frac{\alpha}{2} - (1 - \frac{1}{r}) = 0$, then for $t > 2N$

$$\begin{aligned}
&\int_{\frac{t}{2}}^{t} (\log_e(1+t) - \log_e(1+t-\tau))d\tau \\
&= \frac{t}{2}\log_e(1+t) - \int_{\frac{t}{2}}^{t} \log_e(1+t-\tau)d\tau \\
&= \frac{t}{2}\log_e(1+t) - \int_{0}^{\log_e(1+\frac{t}{2})} se^s ds \\
&= \frac{t}{2}\log_e(1+t) - (1+\frac{t}{2})\log_e(1+\frac{t}{2}) + \int_{0}^{\log_e(1+\frac{t}{2})} e^s ds \\
&\le \frac{t}{2}\log_e(1+t) - \frac{t}{2}\log_e(1+\frac{t}{2}) + (1+\frac{t}{2}) - 1 \\
&= \frac{t}{2}\log_e\frac{2(1+t)}{2+t} + \frac{t}{2} \\
&\le \frac{1}{2}(1+\log_e 2)t. \qquad (2.40)
\end{aligned}$$

If $\frac{\alpha}{2} - (1 - \frac{1}{r}) < 0$, then for $t > 2N$

$$\int_{\frac{t}{2}}^{t} (1+t-\tau)^{\frac{\alpha}{2}-(1-\frac{1}{r})} d\tau \le Ct^{\frac{\alpha}{2}+1-(1-\frac{1}{r})}. \qquad (2.41)$$

Inserting (2.40) and (2.41) into (2.39), we find for $1 \le r \le \infty$ and $t > 2N$

$$\int_{\frac{t}{2}}^{t}\int_{0}^{1} (1+t-\theta\tau)^{\frac{\alpha}{2}-1-(1-\frac{1}{r})}\tau d\theta d\tau \le Ct^{\frac{\alpha}{2}+1-(1-\frac{1}{r})},$$

which implies that (2.38) is valid.

From (2.22), (2.32)–(2.35) and (2.37), we conclude for $0 \le \alpha \le 1$, $1 \le r \le \infty$, $(\alpha, r) \ne (1, 1)$ and large time t

$$\left\| |x|^\alpha \big(u(t) - V(\cdot,t)\cdot \int_0^t \mathcal{F}(\tau)d\tau\big) \right\|_{L^r(\Omega_\delta)} \le Ct^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}\log_e(1+t). \qquad (2.42)$$

On the other hand, for $0 \le \alpha \le 1$, $1 \le r \le \infty$ and large time t

$$
\begin{aligned}
&\Big\||x|^\alpha\Big(u(t) - V(\cdot,t)\cdot\int_0^t \mathcal{F}(\tau)d\tau\Big)\Big\|_{L^r(\Omega\setminus\overline{\Omega_\delta})} \\
&\le C\Big(\|u(t)\|_{L^\infty(\Omega\setminus\overline{\Omega_\delta})} + \|V(\cdot,t)\|_{L^\infty(\Omega\setminus\overline{\Omega_\delta})}\int_0^t |\mathcal{F}(\tau)|d\tau\Big) \\
&\le C\Big(\|u(t)\|_{L^\infty(\Omega)} + \|V(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}\int_0^\infty |\mathcal{F}(\tau)|d\tau\Big) \\
&\le Ct^{-1}.
\end{aligned} \tag{2.43}
$$

Combining (2.42) and (2.43) yields for $0 \le \alpha \le 1$, $1 \le r \le \infty$, $(\alpha, r) \ne (1,1)$ and large time t

$$
\Big\||x|^\alpha\Big(u(t) - V(\cdot,t)\cdot\int_0^t \mathcal{F}(\tau)d\tau\Big)\Big\|_{L^r(\Omega)} \le C\Big(t^{-1} + t^{-\frac{1-\alpha}{2}-(1-\frac{1}{r})}\log_e(1+t)\Big). \qquad \square
$$

Proof of Theorem 1.1. The decay in time of $\|\partial_t u(t)\|_{L^\infty(\Omega)} \le Ct^{-2}$ is given in Lemma 3.3. Together with (2.15), Lemmata 2.5, 2.6, we complete the proof of Theorem 1.1. $\square$

Next we give the proof of Theorem 1.2, which characterizes the asymptotic profile of the solution of (1.1).

Proof of Theorem 1.2. Note that $\int_\Omega a(y)dy = 0$, see (2.31), so by using (2.22), the solution u of (1.1) can be written as follows for $t > 0$

$$
\begin{aligned}
&u(x,t) - V(x,t)\cdot\int_0^t \mathcal{F}(\tau)d\tau + \nabla V(x,t)\cdot\int_0^t\int_\Omega (u\otimes u)(y,\tau)dyd\tau \\
&= \int_\Omega (E_t(x-y) - E_t(x))a(y)dy \\
&\quad - \int_0^t\int_\Omega [\nabla_x V(x-y,t-\tau) - \nabla_x V(x,t-\tau)]\cdot(u\otimes u)(y,\tau)dyd\tau \\
&\quad - \int_0^t\int_\Omega [\nabla V(x,t-\tau) - \nabla V(x,t)]\cdot(u\otimes u)(y,\tau)dyd\tau \\
&\quad - \int_0^t\int_{\partial\Omega} \nabla_x V(x-y,t-\tau)\cdot(T[u,p](y,\tau)\cdot\nu)dS_y d\tau.
\end{aligned} \tag{2.44}
$$

$$\begin{aligned}
&\|\int_{\Omega}(E_t(x-y)-E_t(x))a(y)dy\|_{L^r(\Omega)}\\
&=\|\int_0^1\nabla_x E_t(x-sy)\cdot[y\otimes a(y)]dyds\|_{L^r(\Omega)}\\
&\le\|\nabla E_t\|_{L^r(\mathbb{R}^2)}\||y|a\|_{L^1(\Omega)}\\
&\le Ct^{-\frac{1}{2}-(1-\frac{1}{r})},\ \ 1\le r\le\infty,\ \ t>0;
\end{aligned}\tag{2.45}$$

$$\begin{aligned}
&\|\int_0^t\int_{\Omega}[\nabla_x V(x-y,t-\tau)-\nabla_x V(x,t-\tau)]\cdot(u\otimes u)(y,\tau)dyd\tau\|_{L^r(\Omega)}\\
&\le\|\int_0^t\int_{\Omega}|\nabla_x V(x-y,t-\tau)-\nabla_x V(x,t-\tau)|^{\frac{1}{2}}\\
&\quad\times|\nabla_x V(x-y,t-\tau)-\nabla_x V(x,t-\tau)|^{\frac{1}{2}}|u\otimes u|(y,\tau)dyd\tau\|_{L^r(\Omega)}\\
&\le\Big\|\int_0^1\int_0^t\int_{\Omega}(|\nabla_x V(x-y,t-\tau)|+|\nabla_x V(x,t-\tau)|)^{\frac{1}{2}}\\
&\quad\times|\nabla_x^2 V(x-sy,t-\tau)|^{\frac{1}{2}}|y|^{\frac{1}{2}}|u(y,\tau)|^2dyd\tau ds\Big\|_{L^r(\Omega)}\\
&\le C\int_0^{\frac{t}{2}}\|\nabla V(t-\tau)\|^{\frac{1}{2}}_{L^r(\mathbb{R}^2)}\|\nabla^2V(t-\tau)\|^{\frac{1}{2}}_{L^r(\mathbb{R}^2)}\||y|^{\frac{1}{4}}u(\cdot,\tau)\|^2_{L^2(\Omega)}dy\\
&\quad+C\int_{\frac{t}{2}}^t\|\nabla V(t-\tau)\|^{\frac{1}{2}}_{L^1(\mathbb{R}^2)}\|\nabla^2V(t-\tau)\|^{\frac{1}{2}}_{L^1(\mathbb{R}^2)}\||y|^{\frac{1}{4}}u(\cdot,\tau)\|^2_{L^{2r}(\Omega)}dy\\
&\le C\int_0^{\frac{t}{2}}(t-\tau)^{-\frac{3}{4}-(1-\frac{1}{r})}(1+\tau)^{-\frac{3}{4}}d\tau+C\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{3}{4}}\tau^{\frac{1}{4}-2(1-\frac{1}{2r})}d\tau\\
&\le Ct^{-\frac{1}{2}-(1-\frac{1}{r})},\ \ 1\le r\le\infty,\ \ t>>1;
\end{aligned}\tag{2.46}$$

$$\begin{aligned}
&\|\int_0^t\int_{\Omega}[\nabla V(x,t-\tau)-\nabla V(x,t)]\cdot(u\otimes u)(y,\tau)dyd\tau\|_{L^r(\Omega)}\\
&\le\|\int_0^t\int_{\Omega}|\nabla V(x,t-\tau)-\nabla V(x,t)|^{\frac{3}{4}}\\
&\quad\times|\nabla V(x,t-\tau)-\nabla V(x,t)|^{\frac{1}{4}}|u\otimes u|(y,\tau)dyd\tau\|_{L^r(\Omega)}
\end{aligned}$$

$$\leq \| \int_0^1 \int_0^t \int_\Omega (|\nabla V(x,t-\tau)| + |\nabla V(x,t)|)^{\frac{3}{4}}$$

$$\times |\nabla \partial_t V(x,t-s\tau)|^{\frac{1}{4}} \tau^{\frac{1}{4}} |u(y,\tau)|^2 dy d\tau ds \|_{L^r(\Omega)}$$

$$\leq C \int_0^{\frac{t}{2}} (\|\nabla V(t-\tau)\|_{L^r(\mathbb{R}^2)} + \|\nabla V(t)\|_{L^r(\mathbb{R}^2)})^{\frac{3}{4}}$$

$$\times \|\nabla \partial_t V(t-s_0\tau)\|^{\frac{1}{4}}_{L^r(\mathbb{R}^2)} \tau^{\frac{1}{4}} \|u(\tau)\|^2_{L^2(\Omega)} d\tau$$

$$+ C \int_{\frac{t}{2}}^t (\|\nabla V(t-\tau)\|_{L^1(\mathbb{R}^2)} + \|\nabla V(t)\|_{L^1(\mathbb{R}^2)})^{\frac{3}{4}}$$

$$\times \|\nabla \partial_t V(t-s_0\tau)\|^{\frac{1}{4}}_{L^1(\mathbb{R}^2)} \tau^{\frac{1}{4}} \|u(\tau)\|^2_{L^{2r}(\Omega)} d\tau, \quad s_0 \in (0,1)$$

$$\leq C \int_0^{\frac{t}{2}} [(t-\tau)^{-\frac{1}{2}-(1-\frac{1}{r})} + t^{-\frac{1}{2}-(1-\frac{1}{r})}]^{\frac{3}{4}} (t-s_0\tau)^{(-\frac{3}{2}-(1-\frac{1}{r}))\times\frac{1}{4}} (1+\tau)^{\frac{1}{4}-1} d\tau$$

$$+ C \int_{\frac{t}{2}}^t [(t-\tau)^{-\frac{1}{2}} + t^{-\frac{1}{2}}]^{\frac{3}{4}} (t-s_0\tau)^{-\frac{3}{8}} \tau^{\frac{1}{4}-2(1-\frac{1}{2r})} d\tau$$

$$\leq Ct^{-\frac{1}{2}-(1-\frac{1}{r})} + Ct^{-\frac{3}{4}-(1-\frac{1}{r})} \Big(\int_{\frac{t}{2}}^t ((t-\tau)^{-\frac{3}{4}} + t^{-\frac{3}{4}}) d\tau \Big)^{\frac{1}{2}} \Big(\int_{\frac{t}{2}}^t (t-s_0\tau)^{-\frac{3}{4}} d\tau \Big)^{\frac{1}{2}}$$

$$\leq Ct^{-\frac{1}{2}-(1-\frac{1}{r})}, \quad 1 \leq r \leq \infty, \quad t >> 1; \tag{2.47}$$

$$\| \int_0^t \int_{\partial\Omega} \nabla_x V(x-y,t-\tau) \cdot (T[u,p](y,\tau) \cdot \nu) dS_y d\tau \|_{L^r(\Omega_\delta)}$$

$$\leq \int_0^{\frac{t}{2}} \|\nabla V(\cdot,t-\tau)\|_{L^r(\mathbb{R}^2)} \int_{\partial\Omega} |T[u,p](y,\tau)| dS_y d\tau$$

$$+ \int_{\frac{t}{2}}^t \|\nabla V(\cdot,t-\tau)\|_{L^r(\Omega_\delta)} \int_{\partial\Omega} |T[u,p](y,\tau)| dS_y d\tau$$

$$\leq C \int_0^{\frac{t}{2}} (t-\tau)^{-\frac{1}{2}-(1-\frac{1}{r})} \int_{\partial\Omega} |T[u,p](y,\tau)| dS_y d\tau$$

$$+ C \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{1}{2}-(1-\frac{1}{r})}$$

$$\times (\|\nabla u(\tau)\|_{L^\infty(\Omega)} + \|\nabla^2 u(\tau)\|_{L^2(\Omega)} + \|\nabla p(\tau)\|_{L^2(\Omega)}) d\tau$$

$$\le Ct^{-\frac{1}{2}-(1-\frac{1}{r})} \int_0^\infty |\mathcal{F}(\tau)| d\tau + C \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{1}{2}-(1-\frac{1}{r})} \tau^{-\frac{3}{2}} d\tau$$

$$\le Ct^{-\frac{1}{2}-(1-\frac{1}{r})} + \begin{cases} Ct^{-\frac{3}{2}} & \text{if } 2 < r \le \infty, \\ Ct^{-\frac{3}{2}} \log_e(1+t) & \text{if } r = 2, \\ Ct^{-1-(1-\frac{1}{r})} & \text{if } 1 \le r < 2, \end{cases}$$

$$\le Ct^{-\frac{1}{2}-(1-\frac{1}{r})}, \quad 1 \le r \le \infty, \quad t >> 1. \tag{2.48}$$

From (2.44)–(2.48), we conclude for large time t

$$\left\| u(t) - V(x,t) \cdot \int_0^t \mathcal{F}(\tau) d\tau + \nabla V(x,t) \cdot \int_0^t \int_\Omega (u \otimes u)(y,\tau) dy d\tau \right\|_{L^r(\Omega_\delta)}$$

$$\le Ct^{-\frac{1}{2}-(1-\frac{1}{r})}, \quad 1 \le r \le \infty. \tag{2.49}$$

On the other hand, for $1 \le r \le \infty$ and large time t

$$\left\| u(t) - V(x,t) \cdot \int_0^t \mathcal{F}(\tau) d\tau + \nabla V(x,t) \cdot \int_0^t \int_\Omega (u \otimes u)(y,\tau) dy d\tau \right\|_{L^r(\Omega \backslash \overline{\Omega_\delta})}$$

$$\le C \big(\|u(t) - V(\cdot,t) \cdot \int_0^t \mathcal{F}(\tau) d\tau\|_{L^\infty(\Omega \backslash \overline{\Omega_\delta})}$$

$$+ \|\nabla V(\cdot,t)\|_{L^\infty(\Omega \backslash \overline{\Omega_\delta})} \int_0^t \|u(\tau)\|^2_{L^2(\Omega)} d\tau \big)$$

$$\le C \big(\|\|u(t) - V(\cdot,t) \cdot \int_0^t \mathcal{F}(\tau) d\tau\|_{L^\infty(\Omega)}$$

$$+ \|\nabla V(\cdot,t)\|_{L^\infty(\mathbb{R}^2)} \int_0^t (1+\tau)^{-1} d\tau \big)$$

$$\le C(t^{-1} + t^{-\frac{3}{2}} \log_e(1+t)) \le Ct^{-1}. \tag{2.50}$$

Combining (2.49) and (2.50) yields for large time t

$$\Big\|u(t) - V(x,t)\cdot\int_0^t \mathcal{F}(\tau)d\tau + \nabla V(x,t)\cdot\int_0^t\int_\Omega (u\otimes u)(y,\tau)dyd\tau\Big\|_{L^r(\Omega)}$$
$$\le C(t^{-\frac{1}{2}-(1-\frac{1}{r})} + t^{-1})$$
$$\le \begin{cases} Ct^{-\frac{1}{2}-(1-\frac{1}{r})} & \text{if } 1\le r\le 2,\\ Ct^{-1} & \text{if } 2<r\le\infty,\end{cases}$$

which implies for $1\le r\le 2$,

$$\lim_{t\longrightarrow\infty}\frac{t^{\frac{1}{2}+(1-\frac{1}{r})}}{\log_e(1+t)}\Big\|u(t) - V(x,t)\cdot\int_0^t \mathcal{F}(\tau)d\tau$$
$$+\nabla V(x,t)\cdot\int_0^t\int_\Omega (u\otimes u)(y,\tau)dyd\tau\Big\|_{L^r(\Omega)} = 0. \qquad \square$$

3. Decay estimates of higher-order norms for N-S flows

Lemma 3.1. *Suppose $a\in L^1(\Omega)\bigcap L^2_\sigma(\Omega)\bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1<q\le 2$. Then the solution (u,p) of (1.1), given in Lemma 2.2, satisfies for $t\ge T_1$ with some number $T_1>2$*

$$\|\nabla\partial_t u(t)\|_{L^r(\Omega)}\le Ct^{-\frac{3}{2}-(1-\frac{1}{r})} \quad \textit{for} \quad 1<r\le 2.$$

Proof. It follows from problem (1.1) that $v(t)=\partial_t u(t)$ satisfies

$$\begin{cases} \partial_t v - \Delta v + (v\cdot\nabla u + u\cdot\nabla v) + \nabla\partial_t p = 0 & \text{in } \ \Omega\times(1,\infty),\\ \nabla\cdot v = 0 & \text{in } \ \Omega\times(1,\infty),\\ v(x,t)=0 & \text{on } \ \partial\Omega\times(1,\infty),\\ v(x,1)=\partial_t u(1) & \text{in } \ \Omega.\end{cases} \tag{3.1}$$

Moreover, the solution v of (3.1) can be written as follows for $t>\tau>1$

$$v(t) = e^{-(t-\tau)A}v(\tau) - \int_\tau^t e^{-(t-s)A}P(v(s)\cdot\nabla u(s) + u(s)\cdot\nabla v(s))ds. \tag{3.2}$$

Using (3.2) and Lemmata 2.1, 2.5, we get for $1<r<\infty$ and large time t

$$\|\nabla v(t)\|_{L^r(\Omega)} \le \|\nabla e^{-\frac{t}{2}A}v(\frac{t}{2})\|_{L^r(\Omega)}$$

$$+\int_{\frac{t}{2}}^{t} \|\nabla e^{-(t-s)A} P(v(s)\cdot\nabla u(s)+u(s)\cdot\nabla v(s))\|_{L^r(\Omega)} ds$$

$$\le \begin{cases} Ct^{-\frac{1}{2}}\|v(\frac{t}{2})\|_{L^r(\Omega)} & \text{for } 1<r\le 2,\\ C(1+t^{-\frac{1}{2}})\|v(\frac{t}{2})\|_{L^r(\Omega)} & \text{for } 2<r<\infty, \end{cases}$$

$$+\begin{cases} C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}X_r(s)ds & \text{for } 1<r\le 2,\\ C\int_{\frac{t}{2}}^{t}(1+(t-s)^{-\frac{1}{2}})X_r(s)ds & \text{for } 2<r<\infty, \end{cases}$$

$$\le \begin{cases} Ct^{-\frac{3}{2}-(1-\frac{1}{r})} & \text{for } 1<r\le 2,\\ Ct^{-1-(1-\frac{1}{r})} & \text{for } 2<r<\infty, \end{cases}$$

$$+\begin{cases} Ct^{-2-(1-\frac{1}{r})}+Ct^{-2-(1-\frac{1}{r})}f(t) & \text{for } 1<r\le 2,\\ Ct^{-\frac{3}{2}-(1-\frac{1}{r})}+Ct^{-\frac{3}{2}-(1-\frac{1}{r})}f(t) & \text{for } 2<r<\infty, \end{cases}$$

$$\le \begin{cases} Ct^{-\frac{3}{2}-(1-\frac{1}{r})}+Ct^{-2-(1-\frac{1}{r})}f(t) & \text{for } 1<r\le 2,\\ Ct^{-1-(1-\frac{1}{r})}+Ct^{-\frac{3}{2}-(1-\frac{1}{r})}f(t) & \text{for } 2<r<\infty, \end{cases} \tag{3.3}$$

where $f(t)=\sup_{0<s\le t}[s^{\frac{3}{2}+(1-\frac{1}{r})}\|\nabla v(s)\|_{L^r(\Omega)}]$;

$$X_r(s)=\|\nabla u(s)\|_{L^\infty(\Omega)}\|v(s)\|_{L^r(\Omega)}+\|u(s)\|_{L^\infty(\Omega)}\|\nabla v(s)\|_{L^r(\Omega)}$$
$$\le C(s^{-\frac{5}{2}-(1-\frac{1}{r})}+s^{-1}\|\nabla v(s)\|_{L^r(\Omega)}),\ \ 1<r<\infty;$$

$$\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}X_r(s)ds\le Ct^{-2-(1-\frac{1}{r})}+Ct^{-2-(1-\frac{1}{r})}f(t),\quad 1<r\le 2;$$

and

$$\int_{\frac{t}{2}}^{t}X_r(s)ds\le Ct^{-\frac{3}{2}-(1-\frac{1}{r})}+Ct^{-\frac{3}{2}-(1-\frac{1}{r})}f(t),\quad 2<r<\infty.$$

The estimate (3.3) yields for $1<r\le 2$ and large time t

$$\|\nabla v(t)\|_{L^r(\Omega)}\le Ct^{-\frac{3}{2}-(1-\frac{1}{r})}+C_0t^{-2-(1-\frac{1}{r})}f(t). \tag{3.4}$$

There exists a large number $T_1 > 0$ such that $C_0 T_1^{-\frac{1}{2}} \leq \frac{1}{2}$ in (3.4). It follows from (3.4) that for $1 < r \leq 2$ and $t \geq T_1$

$$t^{-\frac{3}{2}-(1-\frac{1}{r})}\|\nabla v(t)\|_{L^r(\Omega)} \leq C + \frac{1}{2}f(t),$$

then

$$f(t) \leq C + \frac{1}{2}f(t), \quad \text{and so} \quad f(t) \leq 2C,$$

which implies for $1 < r \leq 2$ and $t \geq T_1$

$$\|\nabla\partial_t u(t)\|_{L^r(\Omega)} = \|\nabla v(t)\|_{L^r(\Omega)} \leq Ct^{-\frac{3}{2}-(1-\frac{1}{r})}. \qquad \square$$

Lemma 3.2. *Let the exterior domain Ω be of class C^3, and $a \in L^1(\Omega) \bigcap L^2_\sigma(\Omega) \bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1 < q \leq 2$. Then the strong solution (u,p) of (1.1), given in Lemma 2.2, satisfies for $1 < r \leq 2$ and $t \geq T_1$ (T_1 is from Lemma 3.1)*

$$\|\nabla^3 u(t)\|_{L^r(\Omega)} + \|\nabla^2 p(t)\|_{L^r(\Omega)} \leq Ct^{-1-(1-\frac{1}{r})}.$$

Proof. Applying (2.7) to problem (1.1), we get for $1 < r < \infty$ and $t > 0$

$$\|\nabla^3 u(t)\|_{L^r(\Omega)} + \|\nabla^2 p(t)\|_{L^r(\Omega)} \leq C(\|\partial_t u(t)\|_{W^{1,r}(\Omega)} + \|(u\cdot\nabla u)(t)\|_{W^{1,r}(\Omega)}). \tag{3.5}$$

Using Lemma 2.5, we find for $1 < r < \infty$ and $t \geq T_1$

$$\begin{aligned}\|\nabla(u\cdot\nabla u)(t)\|_{L^r(\Omega)} &\leq C(\|\nabla u(t)\|^2_{L^{2r}(\Omega)} + \|u(t)\|_{L^\infty(\Omega)}\|\nabla^2 u(t)\|_{L^r(\Omega)})\\ &\leq C(t^{-1-2(1-\frac{1}{2r})} + t^{-1-1-(1-\frac{1}{r})}) \leq Ct^{-2-(1-\frac{1}{r})}.\end{aligned} \tag{3.6}$$

Using Lemma 3.1 and (3.6), we obtain for $1 < r \leq 2$ and $t \geq T_1$

$$\begin{aligned}&\|\nabla(u\cdot\nabla u)(t)\|_{L^r(\Omega)} + \|\nabla\partial_t u(t)\|_{L^r(\Omega)}\\ &\leq Ct^{-2-(1-\frac{1}{r})} + Ct^{-\frac{3}{2}-(1-\frac{1}{r})} \leq Ct^{-\frac{3}{2}-(1-\frac{1}{r})}.\end{aligned} \tag{3.7}$$

In addition, using Lemmata 2.5, 3.1 yields for $1 < r < \infty$ and $t \geq T_1$

$$\begin{aligned}&\|\partial_t u(t)\|_{L^r(\Omega)} + \|(u\cdot\nabla u)(t)\|_{L^r(\Omega)}\\ &\leq \|\partial_t u(t)\|_{L^r(\Omega)} + \|u(t)\|_{L^\infty(\Omega)}\|\nabla u(t)\|_{L^r(\Omega)}\\ &\leq Ct^{-1-(1-\frac{1}{r})} + Ct^{-1-\frac{1}{2}-(1-\frac{1}{r})}\\ &\leq Ct^{-1-(1-\frac{1}{r})}.\end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8) yields for $1 < r \le 2$ and $t \ge T_1$

$$\|(u\cdot\nabla u)(t)\|_{W^{1,r}(\Omega)} + \|\partial_t u(t)\|_{W^{1,r}(\Omega)} \le Ct^{-1-(1-\frac{1}{r})}.$$

Whence, together with (3.5), we conclude for $1 < r \le 2$ and $t \ge T_1$

$$\|\nabla^3 u(t)\|_{L^r(\Omega)} + \|\nabla^2 p(t)\|_{L^r(\Omega)} \le Ct^{-1-(1-\frac{1}{r})}. \qquad \square$$

Lemma 3.3. *Assume $a \in L^1(\Omega)\bigcap L^2_\sigma(\Omega)\bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1 < q \le 2$. Then the solution (u,p) of (1.1), which is given in Lemma 2.2, satisfies for $t \ge T_2$ with some number $T_2 > 4T_1$ (T_1 is from Lemma 3.2)*

$$\|\partial_t u(t)\|_{L^\infty(\Omega)} \le Ct^{-2};$$
$$\|\partial_{tt} u(t)\|_{L^r(\Omega)} + \|\nabla^2\partial_t u(t)\|_{L^r(\Omega)} + \|\nabla\partial_t p(t)\|_{L^r(\Omega)} \le Ct^{-2-(1-\frac{1}{r})}, \quad 1 < r \le 2.$$

Proof. Firstly we show that for $0 < \alpha < 1$ and $0 < \delta < 1-\alpha$, $v(t) = \partial_t u(t)$ satisfies for $1 < r \le 2$ and $t > 2$, $h \ge 0$

$$\|A^\alpha v(t+h) - A^\alpha v(t)\|_{L^r(\Omega)} \le C(h^\delta t^{-\alpha-\delta-1-(1-\frac{1}{r})} + h^{1-\alpha} t^{-\frac{5}{2}-(1-\frac{1}{r})}). \tag{3.9}$$

Using (3.2), we find for any $t > 2$, $h \ge 0$

$$\begin{aligned} v(t+h) &= e^{-(t+h-\frac{t}{2})A} v(\frac{t}{2}) - \int_{\frac{t}{2}}^{t+h} e^{-(t+h-s)A} P(u(s)\cdot\nabla v(s) + v(s)\cdot\nabla u(s))ds \\ &= e^{-hA} e^{-\frac{t}{2}A} v(\frac{t}{2}) - (\int_t^{t+h} + \int_{\frac{t}{2}}^{t}) e^{-(t-s)A} e^{-hA} P(u(s)\cdot\nabla v(s) + v(s)\cdot\nabla u(s))ds. \end{aligned}$$

Whence we get for $t > 2$, $h \ge 0$

$$\begin{aligned} v(t+h) - v(t) = {} & (e^{-hA} - I) e^{-\frac{t}{2}A} v(\frac{t}{2}) \\ & - \int_{\frac{t}{2}}^{t} (e^{-hA} - I) e^{-(t-s)A} P(u(s)\cdot\nabla v(s) + v(s)\cdot\nabla u(s))ds \\ & - \int_t^{t+h} e^{-(t+h-s)A} P(u(s)\cdot\nabla v(s) + v(s)\cdot\nabla u(s))ds. \end{aligned} \tag{3.10}$$

Note that the Stokes operator A generates a bounded analytic semigroup $\{e^{-tA}\}_{t\ge0}$ in the exterior domain Ω. Therefore, let $0 \le \beta \le 1$ and $1 < q < \infty$, then $\|A^\beta e^{-tA} w\|_{L^q(\Omega)} \le Ct^{-\beta}\|w\|_{L^q(\Omega)}$ for any $w \in D(A)$. Whence for $1 < r < \infty$ and $\varphi \in D(A^\delta)$

$$\begin{aligned}\|(e^{-hA}-I)\varphi\|_{L^r(\Omega)} &= h\Big\|\int_0^1 A^{1-\delta}e^{-shA}A^{\delta}\varphi ds\Big\|_{L^r(\Omega)} \\ &\le Ch\int_0^1 (sh)^{\delta-1}ds\|A^{\delta}\varphi\|_{L^r(\Omega)} \\ &\le \frac{C}{\delta}h^{\delta}\|A^{\delta}\varphi\|_{L^r(\Omega)}. \end{aligned} \tag{3.11}$$

From (3.10), (3.11) and Lemmata 2.5, 3.1, we conclude for $0<\alpha<1$, $0<\delta<1-\alpha$, $1<r\le 2$ and $t>2T_1$, $h\ge 0$

$$\begin{aligned}
&\|A^{\alpha}v(t+h)-A^{\alpha}v(t)\|_{L^r(\Omega)} \\
&\le \|A^{\alpha}(e^{-hA}-I)e^{-\frac{t}{2}A}v(\frac{t}{2})\|_{L^r(\Omega)} \\
&\quad+\int_t^{t+h}\|A^{\alpha}e^{-(t+h-s)A}P(u(s)\cdot\nabla v(s)+v(s)\cdot\nabla u(s))\|_{L^r(\Omega)}ds \\
&\quad+\int_{\frac{t}{2}}^{t}\|(e^{-hA}-I)A^{\alpha}e^{-(t-s)A}P(u(s)\cdot\nabla v(s)+v(s)\cdot\nabla u(s))\|_{L^r(\Omega)}ds \\
&\le Ch^{\delta}\|A^{\alpha+\delta}e^{-\frac{t}{2}A}v(\frac{t}{2})\|_{L^r(\Omega)} \\
&\quad+C\int_t^{t+h}(t+h-s)^{-\alpha}(\|u(s)\|_{L^\infty(\Omega)}\|\nabla v(s)\|_{L^r(\Omega)}+\|v(s)\|_{L^r(\Omega)}\|\nabla u(s)\|_{L^\infty(\Omega)})ds \\
&\quad+Ch^{\delta}\int_{\frac{t}{2}}^{t}(t-s)^{-\alpha-\delta}(\|u(s)\|_{L^\infty(\Omega)}\|\nabla v(s)\|_{L^r(\Omega)}+\|v(s)\|_{L^r(\Omega)}\|\nabla u(s)\|_{L^\infty(\Omega)})ds \\
&\le Ch^{\delta}t^{-\alpha-\delta}\|v(\frac{t}{2})\|_{L^r(\Omega)}+C\int_t^{t+h}(t+h-s)^{-\alpha}s^{-\frac{5}{2}-(1-\frac{1}{r})}ds \\
&\quad+Ch^{\delta}\int_{\frac{t}{2}}^{t}(t-s)^{-\alpha-\delta}s^{-\frac{5}{2}-(1-\frac{1}{r})}ds \\
&\le Ch^{\delta}t^{-\alpha-\delta-1-(1-\frac{1}{r})}+Ch^{1-\alpha}t^{-\frac{5}{2}-(1-\frac{1}{r})},
\end{aligned}$$

which is (3.9). Using (3.2), and after a direct computation, we find for $t>4$

$$\begin{aligned} Av(t) &= Ae^{-\frac{3t}{4}A}v(\tfrac{t}{4}) - (I - e^{-\frac{t}{2}A})P(u\cdot\nabla v + v\cdot\nabla u)(t) \\ &\quad - \int_{\frac{t}{4}}^{\frac{t}{2}} Ae^{-(t-s)A}P(u\cdot\nabla v + v\cdot\nabla u)(s)ds \\ &\quad - \int_{\frac{t}{2}}^{t} Ae^{-(t-s)A}P((u\cdot\nabla v)(s) - (u\cdot\nabla v)(t) + (v\cdot\nabla u)(s) - (v\cdot\nabla u)(t))ds \\ &= I_1(t) + I_2(t) + I_3(t) + I_4(t). \end{aligned} \tag{3.12}$$

It follows from Lemmata 2.5, 3.1 that for $t > 4T_1$

$$\|I_1(t)\|_{L^r(\Omega)} \le Ct^{-1}\|v(\frac{t}{4})\|_{L^r(\Omega)} \le Ct^{-2-(1-\frac{1}{r})},\ \ 1 < r < \infty; \tag{3.13}$$

$$\begin{aligned} \|I_2(t)\|_{L^r(\Omega)} &\le 2\|P(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\Omega)} \\ &\le C(\|v(t)\|_{L^r(\Omega)}\|\nabla u(t)\|_{L^\infty(\Omega)} + \|u(t)\|_{L^\infty(\Omega)}\|\nabla v(t)\|_{L^r(\Omega)}) \\ &\le Ct^{-\frac{5}{2}-(1-\frac{1}{r})},\ \ 1 < r \le 2; \end{aligned} \tag{3.14}$$

$$\begin{aligned} \|I_3(t)\|_{L^r(\Omega)} &\le C\int_{\frac{t}{4}}^{\frac{t}{2}} (t-s)^{-1}(\|P(v\cdot\nabla u)(s)\|_{L^r(\Omega)} + \|P(u\cdot\nabla v)(s)\|_{L^r(\Omega)})ds \\ &\le C\int_{\frac{t}{4}}^{\frac{t}{2}} (t-s)^{-1}(\|v(s)\|_{L^r(\Omega)}\|\nabla u(s)\|_{L^\infty(\Omega)} + \|u(s)\|_{L^\infty(\Omega)}\|\nabla v(s)\|_{L^r(\Omega)})ds \\ &\le C\int_{\frac{t}{4}}^{\frac{t}{2}} (t-s)^{-1}s^{-\frac{5}{2}-(1-\frac{1}{r})}ds \le Ct^{-\frac{5}{2}-(1-\frac{1}{r})},\ \ 1 < r \le 2. \end{aligned} \tag{3.15}$$

Let $t > 2T_1$ and $\frac{t}{2} \le s \le t$. Then using Lemmata 2.5, 3.1 yields

$$\begin{aligned} \|u(t) - u(s)\|_{L^q(\Omega)} &\le \int_s^t \|\partial_\tau u(\tau)\|_{L^q(\Omega)}d\tau \\ &\le C\int_s^t \tau^{-1-(1-\frac{1}{q})}d\tau \le C(t-s)t^{-1-(1-\frac{1}{q})},\quad 1 < q < \infty; \end{aligned} \tag{3.16}$$

$$\begin{aligned} \|\nabla u(t) - \nabla u(s)\|_{L^q(\Omega)} &\le \int_s^t \|\nabla\partial_\tau u(\tau)\|_{L^q(\Omega)}d\tau \\ &\le C\int_s^t \tau^{-\frac{3}{2}-(1-\frac{1}{q})}d\tau \le C(t-s)t^{-\frac{3}{2}-(1-\frac{1}{q})},\quad 1 < q \le 2. \end{aligned} \tag{3.17}$$

Note that by (2.16) and Lemma 2.4, we have for $t > 2T_1$

$$\begin{aligned}\|v(t)\|_{L^\infty(\Omega)} &= \|\partial_t u(t)\|_{L^\infty(\Omega)} \le C\|\partial_t u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\|\nabla\partial_t u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\\ &\le C\|\partial_t u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\|\nabla^2\partial_t u(t)\|^{\frac{1}{2}}_{L^{\frac{4}{3}}(\Omega)}\\ &\le Ct^{-\frac{7}{8}}\|\nabla^2 v(t)\|^{\frac{1}{2}}_{L^{\frac{4}{3}}(\Omega)}. \end{aligned}\tag{3.18}$$

Let $1 < r \le 2$. Using (2.1), (2.4), (3.9), (3.17) and (3.18), we have for $t > 2T_1$

$$\begin{aligned}\|I_4(t)\|_{L^r(\Omega)} &\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1}(\|Pu(s)\cdot\nabla(v(s)-v(t))\|_{L^r(\Omega)}\\ &+\|Pv(s)\cdot\nabla(u(s)-u(t))\|_{L^r(\Omega)}\\ &+\|P(u(s)-u(t))\cdot\nabla v(t)\|_{L^r(\Omega)}\\ &+\|P(v(s)-v(t))\cdot\nabla u(t)\|_{L^r(\Omega)})ds\\ &\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1}(\|u(s)\|_{L^\infty(\Omega)}\|\nabla v(s)-\nabla v(t)\|_{L^r(\Omega)}\\ &+\|v(s)\|_{L^\infty(\Omega)}\|\nabla u(s)-\nabla u(t)\|_{L^r(\Omega)}\\ &+\|u(s)-u(t)\|_{L^\infty(\Omega)}\|\nabla v(t)\|_{L^r(\Omega)}\\ &+\|v(s)-v(t)\|_{L^\infty(\Omega)}\|\nabla u(t)\|_{L^r(\Omega)})ds\\ &\le C\int_{\frac{t}{2}}^{t}(t-s)^{-1}s^{-1}\|A^{\frac{1}{2}}(v(s)-v(t))\|_{L^r(\Omega)}ds\\ &+C\int_{\frac{t}{2}}^{t}(t-s)^{-1+1}t^{-\frac{7}{8}-\frac{3}{2}-(1-\frac{1}{r})}\|\nabla^2 v(s)\|^{\frac{1}{2}}_{L^{\frac{4}{3}}(\Omega)}ds\\ &+Ct^{-\frac{3}{2}-(1-\frac{1}{r})}\int_{\frac{t}{2}}^{t}(t-s)^{-1}\|A^{\frac{1}{4}}(u(s)-u(t))\|^{\frac{1}{2}}_{L^2(\Omega)}\|A^{\frac{3}{4}}(u(s)-u(t))\|^{\frac{1}{2}}_{L^2(\Omega)}ds\\ &+Ct^{-\frac{1}{2}-(1-\frac{1}{r})}\int_{\frac{t}{2}}^{t}(t-s)^{-1}\|A^{\frac{1}{4}}(v(s)-v(t))\|^{\frac{1}{2}}_{L^2(\Omega)}\|A^{\frac{3}{4}}(v(s)-v(t))\|^{\frac{1}{2}}_{L^2(\Omega)}ds\\ &\le Ct^{-\frac{5}{2}-(1-\frac{1}{r})}+CL^{\frac{1}{2}}_{\frac{4}{3}}(t)t^{-\frac{7}{8}-\frac{3}{2}-(1-\frac{1}{r})}\int_{\frac{t}{2}}^{t}s^{(-2-(1-\frac{3}{4}))\times\frac{1}{2}}ds\end{aligned}$$

$$
\begin{aligned}
&+Ct^{-1}\int_{\frac{t}{2}}^{t}(t-s)^{-1}((t-s)^{\delta_1}s^{-\frac{3}{2}-(1-\frac{1}{r})-\delta_1}+(t-s)^{\frac{1}{2}}s^{-\frac{5}{2}-(1-\frac{1}{r})})ds\\
&+Ct^{-\frac{3}{2}-(1-\frac{1}{r})}\int_{\frac{t}{2}}^{t}(t-s)^{-1}[(t-s)^{\delta_2}s^{-\frac{3}{4}-\delta_2}+(t-s)^{\frac{3}{4}}s^{-2}]^{\frac{1}{2}}\\
&\times[(t-s)^{\delta_3}s^{-\frac{5}{4}-\delta_3}+(t-s)^{\frac{1}{4}}s^{-2}]^{\frac{1}{2}}ds\\
&+Ct^{-\frac{1}{2}-(1-\frac{1}{r})}\int_{\frac{t}{2}}^{t}(t-s)^{-1}[(t-s)^{\delta_2}s^{-\frac{7}{4}-\delta_2}+(t-s)^{\frac{3}{4}}s^{-3}]^{\frac{1}{2}}\\
&\times[(t-s)^{\delta_3}s^{-\frac{9}{4}-\delta_3}+(t-s)^{\frac{1}{4}}s^{-3}]^{\frac{1}{2}}ds\\
&\quad(\text{here}\quad 0<\delta_1<\frac{1}{2},\quad 0<\delta_2<\frac{3}{4},\quad 0<\delta_3<\frac{1}{4})\\
\le\ &Ct^{-\frac{5}{2}-(1-\frac{1}{r})}\big(1+L_{\frac{4}{3}}^{\frac{1}{2}}(t)\big)\\
&+Ct^{-\frac{3}{2}-(1-\frac{1}{r})}\int_{\frac{t}{2}}^{t}(t-s)^{-1}\Big((t-s)^{\frac{1}{2}(\delta_2+\delta_3)}s^{-1-\frac{1}{2}(\delta_2+\delta_3)}\\
&+(t-s)^{\frac{\delta_2}{2}+\frac{1}{8}}s^{-\frac{11}{8}-\frac{\delta_2}{2}}+(t-s)^{\frac{\delta_3}{2}+\frac{3}{8}}s^{-\frac{13}{8}-\frac{\delta_3}{2}}+(t-s)^{\frac{1}{2}}s^{-2}\Big)ds\\
&+Ct^{-\frac{1}{2}-(1-\frac{1}{r})}\int_{\frac{t}{2}}^{t}(t-s)^{-1}\Big((t-s)^{\frac{1}{2}(\delta_2+\delta_3)}s^{-2-\frac{1}{2}(\delta_2+\delta_3)}\\
&+(t-s)^{\frac{\delta_2}{2}+\frac{1}{8}}s^{-\frac{19}{8}-\frac{\delta_2}{2}}+(t-s)^{\frac{\delta_3}{2}+\frac{3}{8}}s^{-\frac{21}{8}-\frac{\delta_3}{2}}+(t-s)^{\frac{1}{2}}s^{-3}\Big)ds\\
\le\ &Ct^{-\frac{5}{2}-(1-\frac{1}{r})}\big(1+L_{\frac{4}{3}}^{\frac{1}{2}}(t)\big)+Ct^{-\frac{3}{2}-(1-\frac{1}{r})}(t^{-1}+t^{-\frac{5}{4}}+t^{-\frac{3}{2}})\\
&+Ct^{-\frac{1}{2}-(1-\frac{1}{r})}(t^{-2}+t^{-\frac{9}{4}}+t^{-\frac{5}{2}})\\
\le\ &Ct^{-\frac{5}{2}-(1-\frac{1}{r})}\big(1+L_{\frac{4}{3}}^{\frac{1}{2}}(t)\big). \qquad (3.19)
\end{aligned}
$$

where $L_q(t)=\sup\limits_{2T_0<s\le t}[s^{2+(1-\frac{1}{q})}\|\nabla^2 v(s)\|_{L^q(\Omega)}]$, $L_{\frac{4}{3}}(t)=L_q(t)\mid_{q=\frac{4}{3}}$, and the decay estimate is used (see [11]): Let $w\in D(A^{\frac{1}{2}})$. Then

$$\|\nabla w\|_{L^q(\Omega)}\le C_q\|A^{\frac{1}{2}}w\|_{L^q(\Omega)},\quad 1<q\le 2.$$

From (3.12)–(3.15) and (3.19), we conclude for $t>4T_1$

$$\|Av(t)\|_{L^r(\Omega)}\le Ct^{-2-(1-\frac{1}{r})}+Ct^{-\frac{5}{2}-(1-\frac{1}{r})}L_{\frac{4}{3}}^{\frac{1}{2}}(t),\quad 1<r\le 2. \qquad (3.20)$$

Using (3.20) and Lemmata 2.5, 3.1, we get for $1 < r \le 2$ and $t > 4T_1$

$$\begin{aligned}
&\|\partial_{tt}u(t)\|_{L^r(\Omega)} = \|\partial_t v(t)\|_{L^r(\Omega)} \\
&\le \|Av(t)\|_{L^r(\Omega)} + \|P(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\Omega)} \\
&\le C(t^{-2-(1-\frac{1}{r})} + t^{-\frac{5}{2}-(1-\frac{1}{r})}L_{\frac{4}{3}}^{\frac{1}{2}}(t) \\
&\quad + \|v(t)\|_{L^{2r}(\Omega)}\|\nabla u(t)\|_{L^{2r}(\Omega)} + \|u(t)\|_{L^\infty(\Omega)}\|\nabla v(t)\|_{L^r(\Omega)}) \\
&\le C(t^{-2-(1-\frac{1}{r})} + t^{-\frac{5}{2}-(1-\frac{1}{r})}L_{\frac{4}{3}}^{\frac{1}{2}}(t) + t^{-1-(1-\frac{1}{2r})-\frac{1}{2}-(1-\frac{1}{2r})} + t^{-1-\frac{3}{2}-(1-\frac{1}{r})}) \\
&\le Ct^{-2-(1-\frac{1}{r})} + Ct^{-\frac{5}{2}-(1-\frac{1}{r})}L_{\frac{4}{3}}^{\frac{1}{2}}(t). \qquad (3.21)
\end{aligned}$$

Applying (2.7) to problem (3.1), using (3.21), we obtain for $1 < r \le 2$ and $t > 4T_1$

$$\begin{aligned}
&\|\nabla^2 v(t)\|_{L^r(\Omega)} + \|\nabla\partial_t p(t)\|_{L^r(\Omega)} \\
&\le C(\|\partial_t v(t)\|_{L^r(\Omega)} + \|P(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\Omega)}) \\
&\le Ct^{-2-(1-\frac{1}{r})} + Ct^{-\frac{5}{2}-(1-\frac{1}{r})}L_{\frac{4}{3}}^{\frac{1}{2}}(t). \qquad (3.22)
\end{aligned}$$

It follows from (3.22) that for $1 < r \le 2$ and $t > 4T_1$

$$L_r(t) \le C + Ct^{-\frac{1}{2}}L_{\frac{4}{3}}^{\frac{1}{2}}(t). \qquad (3.23)$$

By taking $r = \frac{4}{3}$ in (2.23), we find there exists $T_2 > 2T_1$ such that for $t \ge T_2$,

$$L_{\frac{4}{3}}(t) \le C, \quad \text{and then} \quad \|\nabla^2 v(t)\|_{L^{\frac{4}{3}}(\Omega)} \le Ct^{-2-(1-\frac{3}{4})} = Ct^{-\frac{9}{4}}. \qquad (3.24)$$

Inserting (3.24) into (3.18), (3.21), (3.22), respectively, we derive for $1 < r \le 2$ and $t \ge T_2$

$$\|\partial_t u(t)\|_{L^\infty(\Omega)} = \|v(t)\|_{L^\infty(\Omega)} \le Ct^{-\frac{7}{8}}\|\nabla^2 v(t)\|_{L^{\frac{4}{3}}(\Omega)}^{\frac{1}{2}} \le Ct^{-2}, \qquad (3.25)$$

$$\|\nabla^2 v(t)\|_{L^r(\Omega)} + \|\nabla\partial_t p(t)\|_{L^r(\Omega)} \le Ct^{-2-(1-\frac{1}{r})}, \qquad (3.26)$$

and

$$\|\partial_{tt}u(t)\|_{L^r(\Omega)} = \|\partial_t v(t)\|_{L^r(\Omega)} \le Ct^{-2-(1-\frac{1}{r})}. \qquad (3.27)$$

From (3.25)–(3.27), we complete the proof of Lemma 3.3. □

Lemma 3.4. *Suppose the exterior domain Ω is of class C^4, and $a \in L^1(\Omega) \bigcap L^2_\sigma(\Omega) \bigcap W_0^{2-\frac{2}{q},q}(\Omega)$ for some $1 < q \leq 2$. Then the strong solution (u, p) of (1.1), given in Lemma 2.2, satisfies for $1 < r \leq 2$ and $t \geq T_2$ (T_2 is from Lemma 3.3)*

$$\|\nabla^4 u(t)\|_{L^r(\Omega)} + \|\nabla^3 p(t)\|_{L^r(\Omega)} \leq Ct^{-1-(1-\frac{1}{r})}.$$

Proof. Applying (2.7) to problem (1.1′) in Page 14, we obtain for $1 < r < \infty$ and $t > 0$

$$\begin{aligned} &\|\nabla^4 u(t)\|_{L^r(\Omega)} + \|\nabla^3 p(t)\|_{L^r(\Omega)} \\ &\leq C(\|\partial_t u(t)\|_{W^{2,r}(\Omega)} + \|(u \cdot \nabla u)(t)\|_{W^{2,r}(\Omega)}) \\ &\leq C(\|\partial_t u(t)\|_{L^r(\Omega)} + \|(u \cdot \nabla u)(t)\|_{L^r(\Omega)} \\ &\quad + \|\nabla^2 \partial_t u(t)\|_{L^r(\Omega)} + \|\nabla^2 (u \cdot \nabla u)(t)\|_{L^r(\Omega)}). \end{aligned} \tag{3.28}$$

Using Lemmata 2.5, 3.1, 3.2, we get for $1 < r \leq 2$ and $t \geq T_2$

$$\begin{aligned} &\|\nabla^2 (u \cdot \nabla u)(t)\|_{L^r(\Omega)} \\ &\leq C(\|\nabla u(t)\|_{L^\infty(\Omega)} \|\nabla^2 u(t)\|_{L^r(\Omega)} + \|u(t)\|_{L^\infty(\Omega)} \|\nabla^3 u(t)\|_{L^r(\Omega)}) \\ &\leq C(t^{-\frac{3}{2}-1-(1-\frac{1}{r})} + t^{-1-1-(1-\frac{1}{r})}) \leq Ct^{-2-(1-\frac{1}{r})}. \end{aligned} \tag{3.29}$$

From (3.8), (3.28), (3.29) and Lemma 3.3, we infer for $1 < r \leq 2$ and $t \geq T_2$

$$\|\nabla^4 u(t)\|_{L^r(\Omega)} + \|\nabla^3 p(t)\|_{L^r(\Omega)} \leq Ct^{-1-(1-\frac{1}{r})}. \qquad \square$$

Proof of Theorem 1.3. The decay rates of $\|\nabla^{2+k} u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k} p(t)\|_{L^r(\Omega)}$ with $k = 1, 2$ have been established in Lemmata 3.2, 3.4 respectively for large time t. In order to get the decays of $\|\nabla^{2+k} u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k} p(t)\|_{L^r(\Omega)}$ for $k \geq 3$, under the assumption on the exterior domain Ω, which is of class C^k, we appeal the regularity theory (2.7) on the steady Stokes system, and find for $1 < r < \infty$ and $t > 1$

$$\begin{aligned} &\|\nabla^{2+k} u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k} p(t)\|_{L^r(\Omega)} \\ &\leq C(\|\partial_t u(t)\|_{W^{k,r}(\Omega)} + \|(u \cdot \nabla u)(t)\|_{W^{k,r}(\Omega)}) \\ &\leq C(\|\partial_t u(t)\|_{L^r(\Omega)} + \|(u \cdot \nabla u)(t)\|_{L^r(\Omega)} \\ &\quad + \|\nabla^k \partial_t u(t)\|_{L^r(\Omega)} + \|\nabla^k (u \cdot \nabla u)(t)\|_{L^r(\Omega)}), \quad k \geq 3. \end{aligned} \tag{3.30}$$

Applying the regularity estimate (2.7) to problem (3.1), we get for $1 < r < \infty$ and $t > 1$

$$\begin{aligned}
&\|\nabla^k \partial_t u(t)\|_{L^r(\Omega)} = \|\nabla^k v(t)\|_{L^r(\Omega)} \\
&\le C(\|(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{W^{k-2,r}(\Omega)} + \|\partial_t v(t)\|_{W^{k-2,r}(\Omega)}) \\
&\le C(\|(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\Omega)} + \|\partial_t v(t)\|_{L^r(\Omega)} \\
&\quad + \|\nabla^{k-2}(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\Omega)} + \|\nabla^{k-2}\partial_t v(t)\|_{L^r(\Omega)}), \quad k \ge 3.
\end{aligned} \tag{3.31}$$

Combining (3.30) and (3.31), using Lemmata 2.5, 3.1, 3.3, we conclude for $1 < r \le 2$ and large time t

$$\begin{aligned}
&\|\nabla^{2+k} u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k} p(t)\|_{L^r(\Omega)} \\
&\le C(\|\partial_t u(t)\|_{L^r(\Omega)} + \|u(t)\|_{L^{2r}(\Omega)}\|\nabla u(t)\|_{L^{2r}(\Omega)} + \|\nabla u(t)\|_{L^\infty(\Omega)}\|v(t)\|_{L^r(\Omega)} \\
&\quad + \|u(t)\|_{L^\infty(\Omega)}\|\nabla v(t)\|_{L^r(\Omega)} + \|\partial_t v(t)\|_{L^r(\Omega)} + \|\nabla^k(u\cdot\nabla u)(t)\|_{L^r(\Omega)} \\
&\quad + \|\nabla^{k-2}(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\Omega)} + \|\nabla^{k-2}\partial_t v(t)\|_{L^r(\Omega)}) \\
&\le C(t^{-1-(1-\frac{1}{r})} + \|\nabla^k(u\cdot\nabla u)(t)\|_{L^r(\Omega)} \\
&\quad + \|\nabla^{k-2}(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\Omega)} + \|\nabla^{k-2}\partial_t v(t)\|_{L^r(\Omega)}), \quad k \ge 3.
\end{aligned} \tag{3.32}$$

Using the methods employed in the proofs of Lemmata 3.1–3.4, we can establish the decays of $\|\nabla^{2+k}u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k}p(t)\|_{L^r(\Omega)}$ with $k \ge 3$, $1 < r \le 2$.

For example, set $k = 3$ in (3.32). Then by Lemmata 2.5, 3.1–3.4, we have for $1 < r \le 2$ and large time t

$$\begin{aligned}
&\|\nabla^5 u(t)\|_{L^r(\Omega)} + \|\nabla^4 p(t)\|_{L^r(\Omega)} \\
&\le C(t^{-1-(1-\frac{1}{r})} + \|\nabla^3(u\cdot\nabla u)(t)\|_{L^r(\Omega)} \\
&\quad + \|\nabla(v\cdot\nabla u + u\cdot\nabla v)(t)\|_{L^r(\Omega)} + \|\nabla\partial_t v(t)\|_{L^r(\Omega)}) \\
&\le C\|\nabla\partial_t v(t)\|_{L^r(\Omega)} + Ct^{-1-(1-\frac{1}{r})}.
\end{aligned} \tag{3.33}$$

Set $\widetilde{v}(t) = \partial_t v(t)$, where $v(t) = \partial_t u(t)$. Then from problem (3.1), we have

$$\begin{cases}
\partial_t\widetilde{v} - \Delta\widetilde{v} + (\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v}) + \nabla\partial_{tt}p = 0 & \text{in } \Omega\times(1,\infty), \\
\nabla\cdot\widetilde{v} = 0 & \text{in } \Omega\times(1,\infty), \\
\widetilde{v}(x,t) = 0 & \text{on } \partial\Omega\times(1,\infty), \\
\widetilde{v}(x,1) = \partial_{tt}u(1) & \text{in } \Omega.
\end{cases} \tag{3.34}$$

Moreover it holds for $t > 2$

$$\widetilde{v}(t) = e^{-\frac{t}{2}A}\widetilde{v}(\frac{t}{2}) - \int_{\frac{t}{2}}^{t} e^{-(t-s)A}P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})ds. \tag{3.35}$$

Note that for each $1 < r < \infty$, the projection $P: \ L^r(\Omega) \longrightarrow L^r_\sigma(\Omega)$ is bounded. Using (3.35) and Lemmata 2.1, 3.1, 3.3, checking the proof of $\|\nabla v(t)\|_{L^r(\Omega)}$ (see Lemma 3.1), we have for $1 < r \le 2$ and large time t

$$\begin{aligned}
\|\nabla\widetilde{v}(t)\|_{L^r(\Omega)} &\le \|\nabla e^{-\frac{t}{2}A}\widetilde{v}(\frac{t}{2})\|_{L^r(\Omega)} \\
&+ \int_{\frac{t}{2}}^{t} \|\nabla e^{-(t-s)A}P(\widetilde{v}(s)\cdot\nabla u(s) + 2v(s)\cdot\nabla v(s) + u(s)\cdot\nabla\widetilde{v}(s))\|_{L^r(\Omega)}ds \\
&\le Ct^{-\frac{1}{2}}\|\widetilde{v}(\frac{t}{2})\|_{L^r(\Omega)} + C\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\widetilde{X}_r(s)ds \\
&\le Ct^{-\frac{5}{2}-(1-\frac{1}{r})} + Ct^{-3-(1-\frac{1}{r})} + Ct^{-3-(1-\frac{1}{r})}\widetilde{f}(t) \\
&\le Ct^{-\frac{5}{2}-(1-\frac{1}{r})} + Ct^{-3-(1-\frac{1}{r})}\widetilde{f}(t),
\end{aligned} \tag{3.36}$$

$$\widetilde{f}(t) = \sup_{0<s\le t}[s^{\frac{5}{2}+(1-\frac{1}{r})}\|\nabla\widetilde{v}(s)\|_{L^r(\Omega)}],\ 1 < r \le 2,\ t >> 1;$$

$$\begin{aligned}
\widetilde{X}_r(s) &= \|\nabla u(s)\|_{L^\infty(\Omega)}\|\widetilde{v}(s)\|_{L^r(\Omega)} \\
&\quad + 2\|v(s)\|_{L^\infty(\Omega)}\|\nabla v(s)\|_{L^r(\Omega)} + \|u(s)\|_{L^\infty(\Omega)}\|\nabla\widetilde{v}(s)\|_{L^r(\Omega)} \\
&\quad \le C(s^{-\frac{7}{2}-(1-\frac{1}{r})} + s^{-1}\|\nabla\widetilde{v}(s)\|_{L^r(\Omega)}),\ \ 1 < r \le 2,\ \ s >> 1;
\end{aligned}$$

$$\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{1}{2}}\widetilde{X}_r(s)ds \le Ct^{-3-(1-\frac{1}{r})} + Ct^{-3-(1-\frac{1}{r})}\widetilde{f}(t), \quad 1 < r \le 2;$$

The estimate (3.36) yields for $1 < r \le 2$ and large time t

$$t^{\frac{5}{2}+(1-\frac{1}{r})}\|\nabla\widetilde{v}(t)\|_{L^r(\Omega)} \le C + \widetilde{C_0}t^{-\frac{1}{2}}\widetilde{f}(t). \tag{3.37}$$

There exists a large number $\widetilde{T_0}$ such that $\widetilde{C_0}\widetilde{T_0}^{-\frac{1}{2}} \le \frac{1}{2}$ in (3.37). Whence from (3.37), we get for $1 < r \le 2$ and $t \ge \widetilde{T_0}$

$$t^{\frac{5}{2}+(1-\frac{1}{r})}\|\nabla\widetilde{v}(t)\|_{L^r(\Omega)} \le C + \frac{1}{2}\widetilde{f}(t).$$

This shows for $1 < r \leq 2$ and $t \geq \widetilde{T_0}$

$$\widetilde{f}(t) \leq C + \frac{1}{2}\widetilde{f}(t), \quad \text{and then} \quad \widetilde{f}(t) \leq 2C,$$

which implies for $1 < r \leq 2$ and $t \geq \widetilde{T_0}$

$$\|\nabla \partial_t v(t)\|_{L^r(\Omega)} = \|\nabla \widetilde{v}(t)\|_{L^r(\Omega)} \leq Ct^{-\frac{5}{2}-(1-\frac{1}{r})}. \tag{3.38}$$

Inserting (3.38) into (3.33), we obtain for $1 < r \leq 2$ and $t \geq \widetilde{T_0}$

$$\begin{aligned}\|\nabla^5 u(t)\|_{L^r(\Omega)} + \|\nabla^4 p(t)\|_{L^r(\Omega)} &\leq Ct^{-1-(1-\frac{1}{r})} + Ct^{-\frac{5}{2}-(1-\frac{1}{r})} \\ &\leq Ct^{-1-(1-\frac{1}{r})}.\end{aligned} \tag{3.39}$$

Set $k = 3$ in (3.31). Using (3.31), (3.38) and Lemmata 2.5, 3.1, 3.3, we get for $1 < r < 2$ and $t \geq \widetilde{T_0}$

$$\begin{aligned}\|\nabla^3 v(t)\|_{L^r(\Omega)} &\leq C(\|\nabla u(t)\|_{L^\infty(\Omega)}\|v(t)\|_{L^r(\Omega)} + \|u(t)\|_{L^\infty(\Omega)}\|\nabla v(t)\|_{L^r(\Omega)} \\ &\quad + \|\nabla v(t)\|_{L^r(\Omega)}\|\nabla u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^{2r}(\Omega)}\|\nabla^2 u(t)\|_{L^{2r}(\Omega)} \\ &\quad + \|u(t)\|_{L^\infty(\Omega)}\|\nabla^2 v(t)\|_{L^r(\Omega)} + \|\partial_t v(t)\|_{L^r(\Omega)} + \|\nabla \partial_t v(t)\|_{L^r(\Omega)}) \\ &\leq Ct^{-2-(1-\frac{1}{r})}.\end{aligned} \tag{3.40}$$

Take $k = 4$ in (3.32). Then by (3.39), (3.40) and Lemmata 2.5, 3.1–3.4, we have for $1 < r \leq 2$ and $t \geq \widetilde{T_0}$

$$\begin{aligned}&\|\nabla^6 u(t)\|_{L^r(\Omega)} + \|\nabla^5 p(t)\|_{L^r(\Omega)} \\ &\leq C(t^{-1-(1-\frac{1}{r})} + \|\nabla^4(u \cdot \nabla u)(t)\|_{L^r(\Omega)} \\ &\quad + \|\nabla^2(v \cdot \nabla u + u \cdot \nabla v)(t)\|_{L^r(\Omega)} + \|\nabla^2 \partial_t v(t)\|_{L^r(\Omega)}) \\ &\leq C(t^{-1-(1-\frac{1}{r})} + \|u(t)\|_{L^\infty(\Omega)}\|\nabla^5 u(t)\|_{L^r(\Omega)} \\ &\quad + \|\nabla u(t)\|_{L^\infty(\Omega)}\|\nabla^4 u(t)\|_{L^r(\Omega)} + \|\nabla^2 u(t)\|_{L^\infty(\Omega)}\|\nabla^3 u(t)\|_{L^r(\Omega)} \\ &\quad + \|\nabla u(t)\|_{L^\infty(\Omega)}\|\nabla^2 v(t)\|_{L^r(\Omega)} + \|\nabla v(t)\|_{L^r(\Omega)}\|\nabla^2 u(t)\|_{L^\infty(\Omega)} \\ &\quad + \|v(t)\|_{L^\infty(\Omega)}\|\nabla^3 u(t)\|_{L^r(\Omega)} + \|u(t)\|_{L^\infty(\Omega)}\|\nabla^3 v(t)\|_{L^r(\Omega)} \\ &\quad + \|\partial_t v(t)\|_{L^r(\Omega)} + \|\nabla^2 \partial_t v(t)\|_{L^r(\Omega)}) \\ &\leq C\|\nabla^2 \partial_t v(t)\|_{L^r(\Omega)} + Ct^{-1-(1-\frac{1}{r})}.\end{aligned} \tag{3.41}$$

Now we need to deal with the estimate of $\|\nabla^2 \partial_t v(t)\|_{L^r(\Omega)}$ with $1 < r \leq 2$, $t \geq 2\widetilde{T_0}$.

Firstly we show that for $0<\alpha<1$ and $0<\delta<1-\alpha$, $\widetilde{v}(t)=\partial_t v(t)=\partial_{tt}u(t)$ satisfies for $1<r\le 2$ and $t\ge 2\widetilde{T_0}$, $h\ge 0$

$$\|A^\alpha\widetilde{v}(t+h)-A^\alpha\widetilde{v}(t)\|_{L^r(\Omega)}\le C(h^\delta t^{-\alpha-\delta-2-(1-\frac{1}{r})}+h^{1-\alpha}t^{-\frac{7}{2}-(1-\frac{1}{r})}). \tag{3.42}$$

Using (3.35), we find for $t>2\widetilde{T_0}$, $h\ge 0$

$$\widetilde{v}(t+h)=e^{-(t+h-\frac{t}{2})A}\widetilde{v}(\frac{t}{2})-\int_{\frac{t}{2}}^{t+h}e^{-(t+h-s)A}P(\widetilde{v}\cdot\nabla u+2v\cdot\nabla v+u\cdot\nabla\widetilde{v})(s)ds$$
$$=e^{-hA}e^{-\frac{t}{2}A}\widetilde{v}(\frac{t}{2})-(\int_t^{t+h}+\int_{\frac{t}{2}}^{t})e^{-(t-s)A}e^{-hA}P(\widetilde{v}\cdot\nabla u+2v\cdot\nabla v+u\cdot\nabla\widetilde{v})(s)ds.$$

So for $t\ge 2\widetilde{T_0}$, $h\ge 0$

$$\begin{aligned}\widetilde{v}(t+h)-\widetilde{v}(t) &= (e^{-hA}-I)e^{-\frac{t}{2}A}\widetilde{v}(\frac{t}{2})\\ &\quad-\int_{\frac{t}{2}}^{t}(e^{-hA}-I)e^{-(t-s)A}P(\widetilde{v}\cdot\nabla u+2v\cdot\nabla v+u\cdot\nabla\widetilde{v})(s)ds\\ &\quad-\int_t^{t+h}e^{-(t+h-s)A}P(\widetilde{v}\cdot\nabla u+2v\cdot\nabla v+u\cdot\nabla\widetilde{v})(s)ds.\end{aligned}\tag{3.43}$$

By Lemmata 3.1, 3.3, we have for $1<r\le 2$ and $t\ge\widetilde{T_0}$

$$\|(v\cdot\nabla v)(t)\|_{L^r(\Omega)}\le\|v(t)\|_{L^\infty(\Omega)}\|\nabla v(t)\|_{L^r(\Omega)}\le Ct^{-\frac{7}{2}-(1-\frac{1}{r})}. \tag{3.44}$$

Therefore, from (3.11), (3.38), (3.43), (3.44) and Lemmata 2.5, 3.3, we conclude for $0<\alpha<1$, $0<\delta<1-\alpha$, $1<r\le 2$ and $t\ge 2\widetilde{T_0}$, $h\ge 0$

$$\begin{aligned}&\|A^\alpha\widetilde{v}(t+h)-A^\alpha\widetilde{v}(t)\|_{L^r(\Omega)}\\ &\le\|A^\alpha(e^{-hA}-I)e^{-\frac{t}{2}A}\widetilde{v}(\frac{t}{2})\|_{L^r(\Omega)}\\ &\quad+\int_t^{t+h}\|A^\alpha e^{-(t+h-s)A}P(\widetilde{v}\cdot\nabla u+2v\cdot\nabla v+u\cdot\nabla\widetilde{v})(s)\|_{L^r(\Omega)}ds\\ &\quad+\int_{\frac{t}{2}}^{t}\|(e^{-hA}-I)A^\alpha e^{-(t-s)A}P(\widetilde{v}\cdot\nabla u+2v\cdot\nabla v+u\cdot\nabla\widetilde{v})(s)\|_{L^r(\Omega)}ds\end{aligned}$$

$$\le Ch^\delta \|A^{\alpha+\delta} e^{-\frac{t}{2}A}\widetilde{v}(\frac{t}{2})\|_{L^r(\Omega)} + C\int_t^{t+h} (t+h-s)^{-\alpha} Y_r(s)ds$$
$$+ Ch^\delta \int_{\frac{t}{2}}^t (t-s)^{-\alpha-\delta} Y_r(s)ds$$
$$\le Ch^\delta t^{-\alpha-\delta} \|\widetilde{v}(\frac{t}{2})\|_{L^r(\Omega)} + C\int_t^{t+h} (t+h-s)^{-\alpha} s^{-\frac{7}{2}-(1-\frac{1}{r})}ds$$
$$+ Ch^\delta \int_{\frac{t}{2}}^t (t-s)^{-\alpha-\delta} s^{-\frac{7}{2}-(1-\frac{1}{r})}ds$$
$$\le C(h^\delta t^{-\alpha-\delta-2-(1-\frac{1}{r})} + h^{1-\alpha} t^{-\frac{7}{2}-(1-\frac{1}{r})}),$$

which is (3.42). Here we made use of the estimate for $1 < r \le 2$ and $s \ge \widetilde{T_0}$,

$$Y_r(s) = \|u(s)\|_{L^\infty(\Omega)} \|\nabla \widetilde{v}(s)\|_{L^r(\Omega)} + \|(v\cdot\nabla v)(s)\|_{L^r(\Omega)}$$
$$+ \|\widetilde{v}(s)\|_{L^r(\Omega)} \|\nabla u(s)\|_{L^\infty(\Omega)} \le Cs^{-\frac{7}{2}-(1-\frac{1}{r})}. \tag{3.45}$$

Note that for any $t \ge 4\widetilde{T_0}$

$$A\widetilde{v}(t) = Ae^{-\frac{3t}{4}A}\widetilde{v}(\frac{t}{4}) - (I - e^{-\frac{t}{2}A})P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(t)$$
$$- \int_{\frac{t}{4}}^{\frac{t}{2}} Ae^{-(t-s)A} P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(s)ds$$
$$- \int_{\frac{t}{2}}^{t} Ae^{-(t-s)A}\big(P(u\cdot\nabla\widetilde{v})(s) - P(u\cdot\nabla\widetilde{v})(t)$$
$$+ 2P(v\cdot\nabla v)(s) - 2P(v\cdot\nabla v)(t) + P(\widetilde{v}\cdot\nabla u)(s) - P(\widetilde{v}\cdot\nabla u)(t)\big)ds$$
$$= \widetilde{I}_1(t) + \widetilde{I}_2(t) + \widetilde{I}_3(t) + \widetilde{I}_4(t). \tag{3.46}$$

It follows from (3.45) and Lemma 3.3 that for $1 < r \le 2$ and $t \ge 4\widetilde{T_0}$

$$\|\widetilde{I}_1(t)\|_{L^r(\Omega)} \le Ct^{-1}\|\widetilde{v}(\frac{t}{4})\|_{L^r(\Omega)} \le Ct^{-3-(1-\frac{1}{r})}; \tag{3.47}$$

$$\|\widetilde{I}_2(t)\|_{L^r(\Omega)} \le 2\|P(u\cdot\nabla\widetilde{v} + 2v\cdot\nabla v + \widetilde{v}\cdot\nabla u)(t)\|_{L^r(\Omega)}$$
$$\le CY_r(t) \le Ct^{-\frac{7}{2}-(1-\frac{1}{r})}; \tag{3.48}$$

$$\begin{aligned}\|\widetilde{I}_3(t)\|_{L^r(\Omega)} &\le C\int_{\frac{t}{4}}^{\frac{t}{2}}(t-s)^{-1}Y_r(s)ds\\ &\le C\int_{\frac{t}{4}}^{\frac{t}{2}}(t-s)^{-1}s^{-\frac{7}{2}-(1-\frac{1}{r})}ds\\ &\le Ct^{-\frac{7}{2}-(1-\frac{1}{r})}.\end{aligned} \tag{3.49}$$

Let $t \ge 2\widetilde{T_0}$ and $\frac{t}{2} \le s \le t$. Using (3.38) yields for $1 < q \le 2$

$$\begin{aligned}\|\nabla v(t)-\nabla v(s)\|_{L^q(\Omega)} &\le \int_s^t \|\nabla\partial_\tau v(\tau)\|_{L^q(\Omega)}d\tau\\ &\le C\int_s^t \tau^{-\frac{5}{2}-(1-\frac{1}{q})}d\tau\\ &\le C(t-s)t^{-\frac{5}{2}-(1-\frac{1}{q})}.\end{aligned} \tag{3.50}$$

On the other hand, using (3.38) and Lemma 2.4 yields for $2 < q < \infty$

$$\begin{aligned}\|v(t)-v(s)\|_{L^q(\Omega)} &\le \int_s^t \|\partial_\tau v(\tau)\|_{L^q(\Omega)}d\tau\\ &\le C\int_s^t \|\nabla\partial_\tau v(\tau)\|_{L^{\frac{2q}{2+q}}(\Omega)}d\tau\\ &\le C\int_s^t \tau^{-\frac{5}{2}-(1-\frac{2+q}{2q})}d\tau\\ &\le C(t-s)t^{-2-(1-\frac{1}{q})};\end{aligned} \tag{3.51}$$

Using Lemmata 2.4, 3.3, we find for $2 < q < \infty$ and $t \ge s \ge \frac{t}{2} \ge \widetilde{T_0}$

$$\begin{aligned}\|\nabla u(t)-\nabla u(s)\|_{L^q(\Omega)} &\le \int_s^t \|\nabla\partial_\tau u(\tau)\|_{L^q(\Omega)}d\tau \le C\int_s^t \|\nabla^2\partial_\tau u(\tau)\|_{L^{\frac{2q}{2+q}}(\Omega)}d\tau\\ &\le C\int_s^t \tau^{-2-(1-\frac{2+q}{2q})}d\tau \le C(t-s)t^{-\frac{3}{2}-(1-\frac{1}{q})}.\end{aligned} \tag{3.52}$$

For $q = \infty$. Using (2.16), (2.19), Lemmata 3.1, 3.3, and the estimate (3.40), we find for $t \ge s \ge \frac{t}{2} \ge \widetilde{T_0}$

$$\begin{aligned}
&\|\nabla u(t)-\nabla u(s)\|_{L^\infty(\Omega)}\\
&\le \int_s^t \|\nabla\partial_\tau u(\tau)\|_{L^\infty(\Omega)}d\tau \le C\int_s^t \|\nabla\partial_\tau u(\tau)\|^{\frac12}_{L^4(\Omega)}\|\nabla^2\partial_\tau u(\tau)\|^{\frac12}_{L^4(\Omega)}d\tau\\
&\le C\int_s^t (\|\nabla\partial_\tau u(\tau)\|_{L^2(\Omega)}\|\nabla^2\partial_\tau u(\tau)\|_{L^2(\Omega)})^{\frac14}\\
&\quad\times(\|\nabla^2\partial_\tau u(\tau)\|_{L^2(\Omega)}\|\nabla^3\partial_\tau u(\tau)\|_{L^2(\Omega)})^{\frac14}d\tau\\
&\le C\int_s^t \tau^{(-\frac32-(1-\frac12)-2-(1-\frac12))\times\frac14}\tau^{(-2-(1-\frac12)-2-(1-\frac12))\times\frac14}d\tau\\
&\le C\int_s^t \tau^{-\frac{19}{8}}d\tau \le C(t-s)t^{-\frac{19}{8}}.
\end{aligned}\tag{3.53}$$

Combining (3.52) and (3.53), we conclude for $2<q\le\infty$ and $t\ge s\ge\frac t2\ge\widetilde{T_0}$

$$\|\nabla u(t)-\nabla u(s)\|_{L^q(\Omega)}\le\begin{cases} C(t-s)t^{-\frac32-(1-\frac1q)} & \text{if } 2<q<\infty,\\ C(t-s)t^{-\frac{19}{8}} & \text{if } q=\infty.\end{cases}\tag{3.54}$$

Note that $0<\epsilon<\frac1r-\frac12\Leftrightarrow 0<\frac{r}{1-\epsilon r}<2$ for $1<r<2$. Set $1<r<2$, fix such choice of ϵ, then by (3.51) and Lemma 3.1, we get for $t\ge 2\widetilde{T_0}$

$$\begin{aligned}
&\int_{\frac t2}^t (t-s)^{-1}\|P(v(s)-v(t))\cdot\nabla v(t)\|_{L^r(\Omega)}ds\\
&\le C\int_{\frac t2}^t (t-s)^{-1}\|v(s)-v(t)\|_{L^{\frac1\epsilon}(\Omega)}\|\nabla v(t)\|_{L^{\frac{r}{1-\epsilon r}}(\Omega)}\\
&\le C\int_{\frac t2}^t (t-s)^{-1}(t-s)t^{-2-(1-\epsilon)-\frac32-(1-\frac1r+\epsilon)}ds\\
&\le Ct^{-\frac72-(1-\frac1r)}.
\end{aligned}\tag{3.55}$$

In addition, for $r=2$, by (3.38) and Lemmata 3.1, 3.3, we have for $t\ge 2\widetilde{T_0}$

$$\begin{aligned}
&\int_{\frac t2}^t (t-s)^{-1}\|P(v(s)-v(t))\cdot\nabla v(t)\|_{L^2(\Omega)}ds\\
&\le C\int_{\frac t2}^t (t-s)^{-1}\|v(s)-v(t)\|_{L^4(\Omega)}\|\nabla v(t)\|_{L^4(\Omega)}ds
\end{aligned}$$

$$\leq Ct^{-\frac{9}{4}} \int_{\frac{t}{2}}^{t} (t-s)^{-1} \int_{s}^{t} \|\partial_\tau v(\tau)\|_{L^4(\Omega)} d\tau ds$$
$$\leq Ct^{-\frac{9}{4}} \int_{\frac{t}{2}}^{t} (t-s)^{-1} \int_{s}^{t} (\|\partial_\tau v(\tau)\|_{L^2(\Omega)} \|\nabla \partial_\tau v(\tau)\|_{L^2(\Omega)})^{\frac{1}{2}} d\tau ds$$
$$\leq Ct^{-\frac{9}{4}} \int_{\frac{t}{2}}^{t} (t-s)^{-1} \int_{s}^{t} \tau^{(-2-(1-\frac{1}{2})-\frac{5}{2}-(1-\frac{1}{2}))\times\frac{1}{2}} d\tau ds$$
$$\leq Ct^{-4}, \tag{3.56}$$

where the decay estimate is used in the proof of (3.56): Let $t \geq s \geq \frac{t}{2} \geq \widetilde{T_0}$. Then by Lemmata 3.1, 3.3,

$$\|\nabla v(t)\|_{L^4(\Omega)} \leq C(\|\nabla v(t)\|_{L^2(\Omega)} \|\nabla^2 v(t)\|_{L^2(\Omega)})^{\frac{1}{2}}$$
$$\leq Ct^{(-\frac{3}{2}-(1-\frac{1}{2})-2-(1-\frac{1}{2}))\times\frac{1}{2}} = Ct^{-\frac{9}{4}}, \quad t >> 1.$$

Combining (3.55) and (3.56), we get for $1 < r \leq 2$ and $t \geq 2\widetilde{T_0}$

$$\int_{\frac{t}{2}}^{t} (t-s)^{-1} \|P(v(s)-v(t)) \cdot \nabla v(t)\|_{L^2(\Omega)} ds \leq Ct^{-\frac{7}{2}-(1-\frac{1}{r})}. \tag{3.57}$$

Whence, by means of (2.4), (3.38), (3.42), (3.50), (3.51), (3.54), (3.57) and Lemma 2.5, we have for $1 < r \leq 2$ and $t \geq 2\widetilde{T_0}$

$$\|\widetilde{I_4}(t)\|_{L^r(\Omega)} \leq C \int_{\frac{t}{2}}^{t} (t-s)^{-1} (\|Pu(s) \cdot \nabla(\widetilde{v}(s) - \widetilde{v}(t))\|_{L^r(\Omega)}$$
$$+ \|Pv(s) \cdot \nabla(v(s) - v(t))\|_{L^r(\Omega)} + \|P\widetilde{v}(s) \cdot \nabla(u(s) - u(t))\|_{L^r(\Omega)}$$
$$+ \|P(u(s) - u(t)) \cdot \nabla\widetilde{v}(t)\|_{L^r(\Omega)} + \|P(v(s) - v(t)) \cdot \nabla v(t)\|_{L^r(\Omega)}$$
$$+ \|P(\widetilde{v}(s) - \widetilde{v}(t)) \cdot \nabla u(t)\|_{L^r(\Omega)}) ds$$
$$\leq Ct^{-\frac{7}{2}-(1-\frac{1}{r})} + C \int_{\frac{t}{2}}^{t} (t-s)^{-1} (\|u(s)\|_{L^\infty(\Omega)} \|\nabla\widetilde{v}(s) - \nabla\widetilde{v}(t)\|_{L^r(\Omega)}$$
$$+ \|v(s)\|_{L^\infty(\Omega)} \|\nabla v(s) - \nabla v(t)\|_{L^r(\Omega)} + \|\widetilde{v}(s)\|_{L^r(\Omega)} \|\nabla u(s) - \nabla u(t)\|_{L^\infty(\Omega)}$$
$$+ \|u(s) - u(t)\|_{L^\infty(\Omega)} \|\nabla\widetilde{v}(t)\|_{L^r(\Omega)} + \|\widetilde{v}(s) - \widetilde{v}(t)\|_{L^\infty(\Omega)} \|\nabla u(t)\|_{L^r(\Omega)}) ds$$

$$
\begin{aligned}
&\le Ct^{-\frac{7}{2}-(1-\frac{1}{r})} + Ct^{-1}\int_{\frac{t}{2}}^{t}(t-s)^{-1}\|A^{\frac{1}{2}}(\widetilde{v}(s)-\widetilde{v}(t))\|_{L^r(\Omega)}ds \\
&\quad + C\int_{\frac{t}{2}}^{t}(t-s)^{-1+1}(s^{-2-\frac{5}{2}-(1-\frac{1}{r})} + s^{-\frac{19}{8}-2-(1-\frac{1}{r})} + t^{-2-\frac{5}{2}-(1-\frac{1}{r})})ds \\
&\quad + Ct^{-\frac{1}{2}-(1-\frac{1}{r})}\int_{\frac{t}{2}}^{t}(t-s)^{-1}\|A^{\frac{1}{4}}(\widetilde{v}(s)-\widetilde{v}(t))\|_{L^2(\Omega)}^{\frac{1}{2}}\|A^{\frac{3}{4}}(\widetilde{v}(s)-\widetilde{v}(t))\|_{L^2(\Omega)}^{\frac{1}{2}}ds \\
&\le Ct^{-\frac{27}{8}-(1-\frac{1}{r})} \\
&\quad + Ct^{-1}\int_{\frac{t}{2}}^{t}(t-s)^{-1}((t-s)^{\delta_1}s^{-\frac{5}{2}-(1-\frac{1}{r})-\delta_1} + (t-s)^{\frac{1}{2}}s^{-\frac{7}{2}-(1-\frac{1}{r})})ds \\
&\quad + Ct^{-\frac{1}{2}-(1-\frac{1}{r})}\int_{\frac{t}{2}}^{t}(t-s)^{-1}[(t-s)^{\delta_2}s^{-\frac{11}{4}-\delta_2} + (t-s)^{\frac{3}{4}}s^{-4}]^{\frac{1}{2}} \\
&\quad \times [(t-s)^{\delta_3}s^{-\frac{13}{4}-\delta_3} + (t-s)^{\frac{1}{4}}s^{-4}]^{\frac{1}{2}}ds \\
&\qquad (\text{here} \quad 0<\delta_1<\frac{1}{2}, \quad 0<\delta_2<\frac{3}{4}, \quad 0<\delta_3<\frac{1}{4}) \\
&\le Ct^{-\frac{27}{8}-(1-\frac{1}{r})} + Ct^{-\frac{1}{2}-(1-\frac{1}{r})}\int_{\frac{t}{2}}^{t}(t-s)^{-1}((t-s)^{\frac{1}{2}(\delta_2+\delta_3)}s^{-3-\frac{1}{2}(\delta_2+\delta_3)} \\
&\quad + (t-s)^{\frac{\delta_2}{2}+\frac{1}{8}}s^{-\frac{27}{8}-\frac{\delta_2}{2}} + (t-s)^{\frac{\delta_3}{2}+\frac{3}{8}}s^{-\frac{29}{8}-\frac{\delta_3}{2}} + (t-s)^{\frac{1}{2}}s^{-4})ds \\
&\le Ct^{-\frac{27}{8}-(1-\frac{1}{r})} + Ct^{-\frac{1}{2}-(1-\frac{1}{r})}(t^{-3} + t^{-\frac{13}{4}} + t^{-\frac{7}{2}}) \\
&\le Ct^{-\frac{27}{8}-(1-\frac{1}{r})}, \qquad (3.58)
\end{aligned}
$$

From (3.46)–(3.49) and (3.58), we conclude that for $1 < r \le 2$ and $t \ge 4\widetilde{T_0}$

$$
\|A\widetilde{v}(t)\|_{L^r(\Omega)} \le Ct^{-3-(1-\frac{1}{r})}. \qquad (3.59)
$$

Using (3.45) and (3.59), from problem (3.34), we get for $1 < r \le 2$ and $t \ge 4\widetilde{T_0}$

$$
\begin{aligned}
\|\partial_t\widetilde{v}(t)\|_{L^r(\Omega)} &\le \|A\widetilde{v}(t)\|_{L^r(\Omega)} + \|P(\widetilde{v}\cdot\nabla u + 2v\cdot\nabla v + u\cdot\nabla\widetilde{v})(t)\|_{L^r(\Omega)} \\
&\le C(\|A\widetilde{v}(t)\|_{L^r(\Omega)} + Y_r(t)) \\
&\le C(t^{-3-(1-\frac{1}{r})} + t^{-\frac{7}{2}-(1-\frac{1}{r})}) \\
&\le Ct^{-3-(1-\frac{1}{r})}, \qquad (3.60)
\end{aligned}
$$

where the definition of $Y_r(t)$ is given in (3.45).

Applying (2.7) to problem (3.34), using (3.60), we obtain for $1 < r \le 2$ and $t \ge 4\widetilde{T_0}$

$$\begin{aligned}&\|\nabla^2\widetilde{v}(t)\|_{L^r(\Omega)} + \|\nabla\partial_{tt}p(t)\|_{L^r(\Omega)}\\ &\le C(\|\partial_t\widetilde{v}(t)\|_{L^r(\Omega)} + \|P(\widetilde{v}\cdot\nabla u + v\cdot\nabla v + u\cdot\nabla\widetilde{v})(t)\|_{L^r(\Omega)})\\ &\le Ct^{-3-(1-\frac{1}{r})}.\end{aligned} \tag{3.61}$$

Note that $\widetilde{v}(t) = \partial_t v(t)$. From (3.41) and (3.61), we get for $1 < r \le 2$ and $t \ge 4\widetilde{T_0}$

$$\begin{aligned}\|\nabla^6 u(t)\|_{L^r(\Omega)} + \|\nabla^5 p(t)\|_{L^r(\Omega)} &\le C\|\nabla^2\partial_t v(t)\|_{L^r(\Omega)} + Ct^{-1-(1-\frac{1}{r})}\\ &\le Ct^{-1-(1-\frac{1}{r})}.\end{aligned} \tag{3.62}$$

Following the proofs of (3.39), (3.62), it is not difficult to show that for every integer $k \ge 1$, there exists $t_k > 0$ independent of t, such that for $1 < r \le 2$ and $t \ge t_k$

$$\|\nabla^{2+k}u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k}p(t)\|_{L^r(\Omega)} \le Ct^{-1-(1-\frac{1}{r})}. \tag{3.63}$$

By using the inequalities (2.16), (2.19), we get for every integer $k \ge 1$ and $t \ge t_k$,

$$\begin{aligned}\|\nabla^{2+k}u(t)\|_{L^\infty(\Omega)} &\le C\|\nabla^{2+k}u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\|\nabla^{3+k}u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\\ &\le C\big(\|\nabla^{2+k}u(t)\|_{L^2(\Omega)}\|\nabla^{3+k}u(t)\|^2_{L^2(\Omega)}\|\nabla^{4+k}u(t)\|_{L^2(\Omega)}\big)^{\frac{1}{4}}\\ &\le Ct^{-\frac{3}{2}};\end{aligned} \tag{3.64}$$

and

$$\begin{aligned}\|\nabla^{1+k}p(t)\|_{L^\infty(\Omega)} &\le C\|\nabla^{1+k}p(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\|\nabla^{2+k}u(t)\|^{\frac{1}{2}}_{L^4(\Omega)}\\ &\le C\big(\|\nabla^{1+k}p(t)\|_{L^2(\Omega)}\|\nabla^{2+k}p(t)\|^2_{L^2(\Omega)}\|\nabla^{3+k}p(t)\|_{L^2(\Omega)}\big)^{\frac{1}{4}}\\ &\le Ct^{-\frac{3}{2}}.\end{aligned} \tag{3.65}$$

Let $2 < r < \infty$, and $k \ge 1$ be an integer. By using (3.64) and (3.65), we find for $t \ge t_k$

$$\begin{aligned}&\|\nabla^{2+k}u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k}p(t)\|_{L^r(\Omega)}\\ &\le \|\nabla^{2+k}u(t)\|^{1-\frac{2}{r}}_{L^\infty(\Omega)}\|\nabla^{2+k}u(t)\|^{\frac{2}{r}}_{L^2(\Omega)} + \|\nabla^{1+k}p(t)\|^{1-\frac{2}{r}}_{L^\infty(\Omega)}\|\nabla^{1+k}p(t)\|^{\frac{2}{r}}_{L^2(\Omega)}\\ &\le Ct^{-\frac{3}{2}}.\end{aligned} \tag{3.66}$$

From (3.63)–(3.66), we conclude for each integer $k \ge 1$ and all large time t

$$\|\nabla^{2+k}u(t)\|_{L^r(\Omega)} + \|\nabla^{1+k}p(t)\|_{L^r(\Omega)} \le \begin{cases} Ct^{-1-(1-\frac{1}{r})} & \text{if } \ 1 < r \le 2,\\ Ct^{-\frac{3}{2}} & \text{if } \ 2 < r \le \infty.\end{cases} \tag{3.67}$$

Now we prove the case of $r = 1$. Let $k \geq 1$ be an integer. Using (2.22), Lemmata 2.4, 2.5, we deduce for large time t

$$
\begin{aligned}
\|\nabla^{2+k}u(t)\|_{L^1(\Omega_\delta)} \leq C\|\nabla^{2+k}E_t\|_{L^1(\mathbb{R}^2)}\int\limits_{\Omega}|a(y)|dy \\
+ C(\int\limits_0^{\frac{t}{2}} + \int\limits_{\frac{t}{2}}^{t})\int\limits_{\partial\Omega}\|\nabla^{2+k}V(\cdot - y, t-\tau)\|_{L^1(\Omega_\delta)}|T[u,p](y,\tau)|dS_y d\tau \\
+ C\int\limits_0^{\frac{t}{2}}\|\nabla^{3+k}V(\cdot, t-\tau)\|_{L^1(\mathbb{R}^2)}\int\limits_{\Omega}|(u\otimes u)(y,\tau)|dyd\tau \\
+ C\int\limits_{\frac{t}{2}}^{t}\|\nabla^{1+k}V(\cdot - y, t-\tau)\|_{L^1(\Omega_\delta)}(\|u(\tau)\|_{L^\infty(\Omega)}\|\nabla^2 u(\tau)\|_{L^1(\Omega)} \\
+ \|\nabla u(\tau)\|_{L^\infty(\Omega)}\|\nabla u(\tau)\|_{L^1(\Omega)})d\tau \\
\leq Ct^{-\frac{2+k}{2}}\int\limits_{\Omega}|a(y)|dy + Ct^{-\frac{2+k}{2}}\int\limits_0^{\frac{t}{2}}\|T[u,p](\tau)\|_{L^1(\partial\Omega)}d\tau \\
+ C\int\limits_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{2+k}{2}} \\
\times(\|\nabla u(\tau)\|_{L^\infty(\Omega)} + \|\nabla^2 u(\tau)\|_{L^2(\Omega)} + \|\nabla p(\tau)\|_{L^2(\Omega)})d\tau \\
+ Ct^{-\frac{3+k}{2}}\int\limits_0^{\frac{t}{2}}\|u(\tau)\|_{L^2(\Omega)}^2 d\tau \\
+ C\int\limits_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1+k}{2}}(\|u(\tau)\|_{L^\infty(\Omega)}\|\nabla^2 u(\tau)\|_{L^1(\Omega)} \\
+ \|\nabla u(\tau)\|_{L^\infty(\Omega)}\|\nabla u(\tau)\|_{L^1(\Omega)})d\tau \\
\leq Ct^{-\frac{2+k}{2}} + C\int\limits_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{2+k}{2}}\tau^{-\frac{3}{2}}d\tau \\
+ Ct^{-\frac{3+k}{2}}\int\limits_0^{\frac{t}{2}}(1+\tau)^{-2(1-\frac{1}{2})}d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{1+k}{2}}\tau^{-2}d\tau \\
&\le C(t^{-\frac{2+k}{2}}+t^{-\frac{3}{2}}+t^{-\frac{3+k}{2}}\log_e(1+t)) \\
&\quad+\begin{cases} Ct^{-2}\log_e(1+t) & \text{if } k=1, \\ Ct^{-2} & \text{if } k\ge 2, \end{cases} \\
&\le Ct^{-\frac{3}{2}}.
\end{aligned}
\tag{3.68}
$$

On the other hand, using (3.67), we have for each integer $k \ge 1$ and large time t

$$
\begin{aligned}
\|\nabla^{2+k}u(t)\|_{L^1(\Omega\setminus\Omega_\delta)} &\le C\|\nabla^{2+k}u(t)\|_{L^2(\Omega\setminus\Omega_\delta)} \\
&\le C\|\nabla^{2+k}u(t)\|_{L^2(\Omega)} \\
&\le Ct^{-\frac{3}{2}}.
\end{aligned}
\tag{3.69}
$$

Combining (3.68) and (3.69), we deduce for every integer $k \ge 1$ and large time t

$$
\|\nabla^{2+k}u(t)\|_{L^1(\Omega)} \le Ct^{-\frac{3}{2}}. \tag{3.70}
$$

Let $1 < r < 2$. By using (3.67) and (3.70), we get for every integer $k \ge 1$ and large time t

$$
\|\nabla^{2+k}u(t)\|_{L^r(\Omega)} \le \|\nabla^{2+k}u(t)\|_{L^1(\Omega)}^{1-\frac{2}{r}}\|\nabla^{2+k}u(t)\|_{L^2(\Omega)}^{\frac{2}{r}} \le Ct^{-\frac{3}{2}}. \tag{3.71}
$$

It follows from (3.66), (3.70) and (3.71) that for each integer $k \ge 1$ and large time t

$$
\|\nabla^{2+k}u(t)\|_{L^r(\Omega)} \le Ct^{-\frac{3}{2}}, \quad 1 \le r \le \infty. \qquad \square
$$

4. Weighted decays for higher-order derivatives

Thanks to decay results of higher-order spatial derivatives of the solution of (1.1) obtained in Theorem 1.3, we can establish the weighted decay estimates in time of higher-order norms of Navier–Stokes flows of (1.1). To achieve this aim, we make full use of the representation (2.22) of the solution (u,p) of (1.1). Another powerful application in this section is the finiteness of the total net force exerted on the domain boundary $\partial\Omega$ (see Lemma 2.5), that is,

$$
\int_0^\infty \|T[u,p](\tau)\|_{L^1(\partial\Omega)}d\tau \le C. \tag{4.1}
$$

Proof of Theorem 1.4. Recall $\Omega_\delta = \{x \in \Omega;\ dist(x, \partial\Omega) > \delta\}$, $\delta > 0$. Using (2.22), (4.1) and Lemmata 2.2, 2.4, 2.5, we deduce that for all $1 \le q \le \infty$, $0 < \alpha < 1$ and large time t

$$
\begin{aligned}
\||x|^\alpha \nabla u(t)\|_{L^q(\Omega_\delta)} &\le C\||\cdot|^\alpha \nabla E_t(\cdot)\|_{L^q(\mathbb{R}^2)} \int_\Omega |a(y)|dy \\
&+ C\|\nabla E_t\|_{L^q(\mathbb{R}^2)} \int_\Omega |y|^\alpha |a(y)|dy \\
&+ C(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t)\||\cdot|^\alpha \nabla V(\cdot, t-\tau)\|_{L^q(\Omega_\delta)} \int_{\partial\Omega} |T[u,p]|(y,\tau)dS_y d\tau \\
&+ C(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t) \int_{\partial\Omega} \|\nabla V(\cdot - y, t-\tau)\|_{L^q(\Omega_\delta)} |y|^\alpha |T[u,p]|(y,\tau)dS_y d\tau \\
&+ C\int_0^{\frac{t}{2}} \||\cdot|^\alpha \nabla^2 V(\cdot, t-\tau)\|_{L^q(\mathbb{R}^2)} \int_\Omega |(u\otimes u)(y,\tau)|dy d\tau \\
&+ C\int_0^{\frac{t}{2}} \|\nabla^2 V(\cdot, t-\tau)\|_{L^q(\mathbb{R}^2)} \int_\Omega |y|^\alpha |(u\otimes u)(y,\tau)|dy d\tau \\
&+ C\int_{\frac{t}{2}}^t \||\cdot|^\alpha \nabla V(\cdot, t-\tau)\|_{L^1(\mathbb{R}^2)} \|u(\tau)\|_{L^\infty(\Omega)} \|\nabla u(\tau)\|_{L^q(\Omega)} d\tau \\
&+ C\int_{\frac{t}{2}}^t \|\nabla V(\cdot, t-\tau)\|_{L^1(\mathbb{R}^2)} \|u(\tau)\|_{L^\infty(\Omega)} \||y|^\alpha \nabla u(\tau)\|_{L^q(\Omega)} d\tau \\
&\le Ct^{\frac{\alpha}{2} - \frac{1}{2} - (1-\frac{1}{q})} \int_\Omega |a(y)|dy + Ct^{-\frac{1}{2} - (1-\frac{1}{q})} \int_\Omega |y|^\alpha |a(y)|dy \\
&+ C_\delta (t^{\frac{\alpha}{2} - \frac{1}{2} - (1-\frac{1}{q})} + t^{-\frac{1}{2} - (1-\frac{1}{q})}) \int_0^{\frac{t}{2}} \int_{\partial\Omega} |T[u,p]|(y,\tau)dS_y d\tau \\
&+ C_\delta \int_{\frac{t}{2}}^t ((1+t-\tau)^{\frac{\alpha}{2} - \frac{1}{2} - (1-\frac{1}{q})} + (1+t-\tau)^{-\frac{1}{2} - (1-\frac{1}{q})}) \\
&\times (\|\nabla u(\tau)\|_{L^\infty(\Omega)} + \|\nabla^2 u(\tau)\|_{L^2(\Omega)} + \|\nabla p(\tau)\|_{L^2(\Omega)}) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ Ct^{-1-(1-\frac{1}{q})}\int\limits_0^{\frac{t}{2}} \|u(\tau)\|_{L^2(\Omega)}\||y|^\alpha u(\tau)\|_{L^2(\Omega)}d\tau \\
&+ Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{q})}\int\limits_0^{\frac{t}{2}} \|u(\tau)\|^2_{L^2(\Omega)}d\tau \\
&+ C\int\limits_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}}\|u(\tau)\|_{L^\infty(\Omega)}\||y|^\alpha\nabla u(\tau)\|_{L^q(\Omega)}d\tau \\
&+ C\int\limits_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}+\frac{\alpha}{2}}\|u(\tau)\|_{L^\infty(\Omega)}\|\nabla u(\tau)\|_{L^q(\Omega)}d\tau \\
\le\ & Ct^{\frac{\alpha}{2}-\frac{1}{2}-(1-\frac{1}{q})} + C\int\limits_{\frac{t}{2}}^{t}(1+t-\tau)^{\frac{\alpha}{2}-\frac{1}{2}-(1-\frac{1}{q})}\tau^{-\frac{3}{2}}d\tau \\
&+ Ct^{-1-(1-\frac{1}{q})}\int\limits_0^{\frac{t}{2}}(1+\tau)^{-(1-\frac{1}{2})+\frac{\alpha}{2}-(1-\frac{1}{2})}d\tau \\
&+ Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{q})}\int\limits_0^{\frac{t}{2}}(1+\tau)^{-2(1-\frac{1}{2})}d\tau \\
&+ Cg_q(t)\int\limits_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}}\tau^{-1+\frac{\alpha}{2}-\frac{1}{2}-(1-\frac{1}{q})}d\tau \\
&+ C\int\limits_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}+\frac{\alpha}{2}}\tau^{-1-\frac{1}{2}-(1-\frac{1}{q})}d\tau \\
\le\ & Ct^{\frac{\alpha}{2}-\frac{1}{2}-(1-\frac{1}{q})} + Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{q})}(1+\log_e(1+t)) + Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{q})}g_q(t) \\
\le\ & Ct^{\frac{\alpha}{2}-\frac{1}{2}-(1-\frac{1}{q})} + Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{q})}g_q(t),
\end{aligned} \tag{4.2}
$$

where $g_q(t) = \sup\limits_{0<\tau\le t}[\tau^{-\frac{\alpha}{2}+\frac{1}{2}+(1-\frac{1}{q})}\||x|^\alpha\nabla u(\tau)\|_{L^q(\Omega)}]$ with $1 \le q \le \infty$, $0 < \alpha < 1$. In the proof of (4.2), the following estimates are utilized for $1 \le q \le \infty$ and $t \ge 2$

$$
\int\limits_{\frac{t}{2}}^{t}(1+t-\tau)^{\frac{\alpha}{2}-\frac{1}{2}-(1-\frac{1}{q})}\tau^{-\frac{3}{2}}d\tau
$$

$$\leq Ct^{-\frac{3}{2}}\begin{cases} t^{\frac{\alpha}{2}+\frac{1}{2}-(1-\frac{1}{q})} & \text{if } \frac{\alpha}{2}+\frac{1}{2}-(1-\frac{1}{q})>0,\\ \log_e(1+t) & \text{if } \frac{\alpha}{2}+\frac{1}{2}-(1-\frac{1}{q})=0,\\ 1 & \text{if } \frac{\alpha}{2}+\frac{1}{2}-(1-\frac{1}{q})<0, \end{cases}$$
$$\leq Ct^{\frac{\alpha}{2}-\frac{1}{2}-(1-\frac{1}{q})};$$

and $\||y|^\alpha u(t)\|_{L^2(\Omega)} \leq C(1+t)^{\frac{\alpha}{2}-\frac{1}{2}}$, $0<\alpha<1$, for $t>0$, see [23].

Let $1 \leq q \leq \infty$ and $0<\alpha<1$. It follows from Lemma 2.5 that for all large time t

$$\||x|^\alpha \nabla u(t)\|_{L^q(\Omega\backslash\Omega_\delta)} \leq C\|\nabla u(t)\|_{L^q(\Omega\backslash\Omega_\delta)} \leq C\|\nabla u(t)\|_{L^q(\Omega)} \leq Ct^{-\frac{1}{2}-(1-\frac{1}{q})}. \quad (4.3)$$

Combining (4.2) and (4.3), we conclude for $1 \leq q \leq \infty$, $0<\alpha<1$ and large time t

$$t^{-\frac{\alpha}{2}+\frac{1}{2}+(1-\frac{1}{q})}\||x|^\alpha \nabla u(t)\|_{L^q(\Omega)} \leq C + C_0(q)t^{-\frac{1}{2}}g_q(t), \quad (4.4)$$

There exists a large number $\bar{t}_0$ such that $C_0(q)\bar{t}_0^{-\frac{1}{2}}g_2(t) \leq \frac{1}{2}$. Whence for $t \geq \bar{t}_0$

$$t^{-\frac{\alpha}{2}+\frac{1}{2}+(1-\frac{1}{q})}\||x|^\alpha \nabla u(t)\|_{L^q(\Omega)} \leq C + \frac{1}{2}g_q(t), \quad (4.5)$$

the estimate (4.5) yields for $t \geq \bar{t}_0$

$$g_q(t) \leq C + \frac{1}{2}g_q(t), \text{ and then } g_q(t) \leq 2C. \quad (4.6)$$

From (4.4) and (4.6), we conclude for $1 \leq q \leq \infty$, $0<\alpha<1$ and $t \geq \bar{t}_0$

$$\||x|^\alpha \nabla u(t)\|_{L^q(\Omega)} \leq Ct^{\frac{\alpha}{2}-\frac{1}{2}-(1-\frac{1}{q})},$$

which implies that (1.11) is valid.

Let $k \geq 0$ be an integer, $1 \leq q \leq \infty$, $0<\alpha<1$. Using (1.11), (2.22), (4.1) and Lemmata 2.2, 2.4, 2.5, we deduce for large time t

$$\begin{aligned}
&\||x|^\alpha \nabla^{2+k} u(t)\|_{L^q(\Omega_\delta)}\\
&\leq C\||\cdot|^\alpha \nabla^{2+k}E_t(\cdot)\|_{L^q(\mathbb{R}^2)}\int_\Omega |a(y)|dy + C\|\nabla^{2+k}E_t\|_{L^q(\mathbb{R}^2)}\int_\Omega |y|^\alpha |a(y)|dy\\
&+ C\Big(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\Big)\int_{\partial\Omega} \||\cdot - y|^\alpha \nabla^{2+k}V(\cdot - y, t-\tau)\|_{L^q(\Omega_\delta)}|T[u,p]|(y,\tau)dS_y d\tau\\
&+ C\Big(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\Big)\int_{\partial\Omega} \|\nabla^{2+k}V(\cdot - y, t-\tau)\|_{L^q(\Omega_\delta)}|y|^\alpha |T[u,p]|(y,\tau)dS_y d\tau
\end{aligned}$$

$$
\begin{aligned}
&+ C\int_0^{\frac{t}{2}} \||\cdot - y|^\alpha \nabla^{3+k} V(\cdot, t-\tau)\|_{L^q(\mathbb{R}^2)} \int_\Omega |(u\otimes u)(y,\tau)| dy d\tau \\
&+ C\int_0^{\frac{t}{2}} \|\nabla^{3+k} V(\cdot, t-\tau)\|_{L^q(\mathbb{R}^2)} \int_\Omega |y|^\alpha |(u\otimes u)(y,\tau)| dy d\tau \\
&+ C\int_{\frac{t}{2}}^{t} \||\cdot - y|^\alpha \nabla^{2+k} V(\cdot - y, t-\tau)\|_{L^1(\Omega_\delta)} \|u(\tau)\|_{L^\infty(\Omega)} \|\nabla u(\tau)\|_{L^q(\Omega)} d\tau \\
&+ C\int_{\frac{t}{2}}^{t} \|\nabla^{2+k} V(\cdot - y, t-\tau)\|_{L^1(\Omega_\delta)} \|u(\tau)\|_{L^\infty(\Omega)} \||y|^\alpha \nabla u(\tau)\|_{L^q(\Omega)} d\tau \\
\leq\ & Ct^{\frac{\alpha}{2} - \frac{2+k}{2} - (1-\frac{1}{q})} \int_\Omega |a(y)| dy + Ct^{-\frac{2+k}{2} - (1-\frac{1}{q})} \int_\Omega |y|^\alpha |a(y)| dy \\
&+ C(t^{\frac{\alpha}{2} - \frac{2+k}{2} - (1-\frac{1}{q})} + t^{-\frac{2+k}{2} - (1-\frac{1}{q})}) \int_0^{\frac{t}{2}} \|T[u,p](\tau)\|_{L^1(\partial\Omega)} d\tau \\
&+ C\int_{\frac{t}{2}}^{t} ((1+t-\tau)^{\frac{\alpha}{2} - \frac{2+k}{2} - (1-\frac{1}{q})} + (1+t-\tau)^{-\frac{2+k}{2} - (1-\frac{1}{q})}) \\
&\times (\|\nabla u(\tau)\|_{L^\infty(\Omega)} + \|\nabla^2 u(\tau)\|_{L^2(\Omega)} + \|\nabla p(\tau)\|_{L^2(\Omega)}) d\tau \\
&+ Ct^{-\frac{3+k}{2} - (1-\frac{1}{q})} \int_0^{\frac{t}{2}} \|u(\tau)\|_{L^2(\Omega)} \||y|^\alpha u(\tau)\|_{L^2(\Omega)} d\tau \\
&+ Ct^{\frac{\alpha}{2} - \frac{3+k}{2} - (1-\frac{1}{q})} \int_0^{\frac{t}{2}} \|u(\tau)\|_{L^2(\Omega)}^2 d\tau \\
&+ C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{2+k}{2}} \|u(\tau)\|_{L^\infty(\Omega)} \||y|^\alpha \nabla u(\tau)\|_{L^q(\Omega)} d\tau \\
&+ C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{2+k}{2} + \frac{\alpha}{2}} \|u(\tau)\|_{L^\infty(\Omega)} \|\nabla u(\tau)\|_{L^q(\Omega)} d\tau \\
\leq\ & Ct^{\frac{\alpha}{2} - \frac{2+k}{2} - (1-\frac{1}{q})} + C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{\frac{\alpha}{2} - \frac{2+k}{2} - (1-\frac{1}{q})} \tau^{-\frac{3}{2}} d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+ Ct^{-\frac{3+k}{2}-(1-\frac{1}{q})} \int_0^{\frac{t}{2}} (1+\tau)^{\frac{\alpha}{2}-2(1-\frac{1}{2})} d\tau \\
&+ Ct^{\frac{\alpha}{2}-\frac{3+k}{2}-(1-\frac{1}{q})} \int_0^{\frac{t}{2}} (1+\tau)^{-2(1-\frac{1}{2})} d\tau \\
&+ C \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{2+k}{2}} \tau^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})} d\tau \\
&+ C \int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{2+k}{2}+\frac{\alpha}{2}} \tau^{-\frac{3}{2}-(1-\frac{1}{q})} d\tau \\
\le\ & Ct^{\frac{\alpha}{2}-\frac{2+k}{2}-(1-\frac{1}{q})} + +Ct^{\frac{\alpha}{2}-\frac{3+k}{2}-(1-\frac{1}{q})}(1+\log_e(1+t)) \\
&+ \begin{cases} Ct^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})} & \text{if } k=0 \text{ and } \ell(\alpha,q)>0, \\ Ct^{-\frac{3}{2}}\log_e(1+t) & \text{if } k=0 \text{ and } \ell(\alpha,q)=0, \\ Ct^{-\frac{3}{2}} & \text{if } k=0 \text{ and } \ell(\alpha,q)<0, \\ Ct^{-\frac{3}{2}} & \text{if } k\ge 1, \end{cases} \\
&+ \begin{cases} Ct^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})}\log_e(1+t) & \text{if } k=0, \\ Ct^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})} & \text{if } k\ge 1, \end{cases} \\
&+ \begin{cases} Ct^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})} & \text{if } k=0, \\ Ct^{-\frac{3}{2}-(1-\frac{1}{q})} & \text{if } k\ge 1, \end{cases} \\
\le\ & \begin{cases} Ct^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})} & \text{if } k=0 \text{ and } \ell(\alpha,q)>0, \\ Ct^{-\frac{3}{2}}\log_e(1+t) & \text{if } k=0 \text{ and } \ell(\alpha,q)=0, \\ Ct^{-\frac{3}{2}} & \text{if } k=0 \text{ and } \ell(\alpha,q)<0, \\ Ct^{-\frac{3}{2}} & \text{if } k\ge 1, \end{cases} \\
&+ \begin{cases} Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{q})} & \text{if } k=0, \\ Ct^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})} & \text{if } k\ge 1, \end{cases}
\end{aligned} \tag{4.7}
$$

where $\ell(\alpha, q) = \frac{\alpha}{2} - (1 - \frac{1}{q})$. In the proof of (4.7), the decay estimate: $\||y|^\alpha u(t)\|_{L^2(\Omega)} \le C(1+t)^{\frac{\alpha}{2}-\frac{1}{2}}$, $0 < \alpha < 1$, for $t > 0$ is employed, see [23].

It follows from (4.7) with $k = 0$ that for all $1 \le q \le \infty$, $0 < \alpha < 1$ and large time t

$$
\||x|^\alpha \nabla^2 u(t)\|_{L^q(\Omega_\delta)} \le Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{q})} + \begin{cases} Ct^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})} & \text{if } \ell(\alpha,q)>0, \\ Ct^{-\frac{3}{2}}\log_e(1+t) & \text{if } \ell(\alpha,q)=0, \\ Ct^{-\frac{3}{2}} & \text{if } \ell(\alpha,q)<0. \end{cases} \tag{4.8}
$$

Recall the estimate (1.3), namely, it holds for large time t

$$\|\nabla^2 u(t)\|_{L^q(\Omega)} \le \begin{cases} Ct^{-1-(1-\frac{1}{q})} & \text{if } 1 \le q < \infty, \\ Ct^{-\frac{7}{4}} & \text{if } q = \infty. \end{cases} \tag{4.9}$$

Note that it easily verify that $q \neq \infty$ if $\ell(\alpha, q) = \frac{\alpha}{2} - (1 - \frac{1}{q}) \ge 0$ with $0 < \alpha < 1$. Whence combining (4.8) and (4.9), we conclude for all $1 \le q \le \infty$, $\ell(\alpha, q) \ge 0$, $0 < \alpha < 1$ and large time t

$$\begin{aligned} \||x|^\alpha \nabla^2 u(t)\|_{L^q(\Omega)} &= \||x|^\alpha \nabla^2 u(t)\|_{L^q(\Omega\setminus\Omega_\delta)} + \||x|^\alpha \nabla^2 u(t)\|_{L^q(\Omega_\delta)} \\ &\le C\|\nabla^2 u(t)\|_{L^q(\Omega\setminus\Omega_\delta)} + \||x|^\alpha \nabla^2 u(t)\|_{L^q(\Omega_\delta)} \\ &\le Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{q})} + \begin{cases} Ct^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})} & \text{if } \ell(\alpha,q) > 0, \\ Ct^{-\frac{3}{2}} \log_e(1+t) & \text{if } \ell(\alpha,q) = 0, \end{cases} \\ &\le Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{q})}. \end{aligned} \tag{4.10}$$

If $\ell(\alpha, q) = \frac{\alpha}{2} - (1 - \frac{1}{q}) < 0$. From (4.7) with $k = 0$, we conclude for $1 \le q \le \infty$, $0 < \alpha < 1$ and large time t

$$\||x|^\alpha \nabla^2 u(t)\|_{L^q(\Omega_\delta)} \le C(t^{-\frac{3}{2}} + t^{\frac{\alpha}{2}-1-(1-\frac{1}{q})}). \tag{4.11}$$

Combining (4.9) and (4.11), we obtain for $1 \le q \le \infty$, $0 < \alpha < 1$, $\ell(\alpha, q) < 0$ and large time t

$$\begin{aligned} \||x|^\alpha \nabla^2 u(t)\|_{L^q(\Omega)} &\le \begin{cases} C(t^{-\frac{3}{2}} + t^{\frac{\alpha}{2}-1-(1-\frac{1}{q})}) & \text{if } 1 \le q < \infty, \\ Ct^{-\frac{3}{2}} & \text{if } q = \infty. \end{cases} \\ &\le \Big\{ C(t^{-\frac{3}{2}} + t^{\frac{\alpha}{2}-1-(1-\frac{1}{q})}). \end{aligned} \tag{4.12}$$

It follows from (4.10) and (4.12) that for $1 \le q \le \infty$, $0 < \alpha < 1$ and large time t

$$\||x|^\alpha \nabla^2 u(t)\|_{L^q(\Omega)} \le \begin{cases} Ct^{\frac{\alpha}{2}-1-(1-\frac{1}{q})} & \text{if } \ell(\alpha,q) \ge 0, \\ C(t^{-\frac{3}{2}} + t^{\frac{\alpha}{2}-1-(1-\frac{1}{q})}) & \text{if } \ell(\alpha,q) < 0, \end{cases}$$

which is (1.12).

By means of (4.7), we find for $1 \le q \le \infty$, $0 < \alpha < 1$, $k \ge 1$ and large time t

$$\||x|^\alpha \nabla^{2+k} u(t)\|_{L^q(\Omega_\delta)} \le C(t^{-\frac{3}{2}} + t^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})}). \tag{4.13}$$

In addition, by Theorem 1.3, we have for all $1 \le q \le \infty$, $0 < \alpha < 1$, $k \ge 1$ and large time t

$$\begin{aligned}\||x|^\alpha \nabla^{2+k} u(t)\|_{L^q(\Omega\setminus\Omega_\delta)} &\le C\|\nabla^{2+k} u(t)\|_{L^q(\Omega\setminus\Omega_\delta)}\\ &\le C\|\nabla^{2+k} u(t)\|_{L^q(\Omega)}\\ &\le Ct^{-\frac{3}{2}} \qquad (4.14)\end{aligned}$$

Combining (4.13) and (4.14) yields for $1 \le q \le \infty$, $0 < \alpha < 1$, $k \ge 1$ and large time t

$$\||x|^\alpha \nabla^{2+k} u(t)\|_{L^q(\Omega)} \le C(t^{-\frac{3}{2}} + t^{\frac{\alpha}{2}-\frac{3}{2}-(1-\frac{1}{q})}),$$

which is (1.13). □

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