Convergence of HX Preconditioner for Maxwell's Equations with Jump Coefficients (ii): The Main Results

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Abstract

This paper is the second one of two serial articles, whose goal is to prove convergence of HX Preconditioner (proposed by Hiptmair and Xu [17]) for Maxwell's equations with jump coefficients. In this paper, based on the auxiliary results developed in the first paper [18], we establish a new regular Helmholtz decomposition for edge finite element functions in three dimensions, which is nearly stable with respect to a weight function. By using this Helmholtz decomposition, we give an analysis of the convergence of the HX preconditioner for the case with strongly discontinuous coefficients. We show that the HX preconditioner possesses fast convergence, which not only is nearly optimal with respect to the finite element mesh size but also is independent of the jumps in the coefficients across the interface between two neighboring subdomains.

Key Words. Maxwell's equations, discontinuous coefficients, Nedelec elements, regular Helmholtz decomposition, HX preconditioner, convergence

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1 Introduction

Consider the following Maxwell's equations ([4, 8, 25, 30, 33]):

$$\begin{aligned} \mathbf{curl}(\alpha\, \mathbf{curl}\, \mathbf{u}) + \beta \mathbf{u} &= \mathbf{f} \quad \text{in} \quad \Omega, \\ \mathbf{u} \times \mathbf{n} &= \mathbf{0} \quad \text{on} \quad \Omega, \end{aligned} \tag{1.1}$$

where Ω is a simply-connected bounded domain in $\mathbf{R}^3$ with boundary $\partial\Omega$, occupied often by nonhomogeneous medium; $\mathbf{f}$ is a vector field in $(L^2(\Omega))^3$. The coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are two positive functions, which may have jumps across the interface between two neighboring different media in Ω. The problem (1.1) arises in different applications, for instance, in the eddy current model in computational electromagnetics [4].

The Nedelec edge finite element method (see [27]) is a popular discretization method of the equations (1.1), and the resulting algebraic system is in general needed to be solved by some preconditioned iterative method. As pointed out in [12], the construction of an efficient preconditioner for the resulting system is much more difficult than that for the standard elliptic equations of the second order. There are some works to construct such efficient preconditioners in literature, see, for example, [2] [12] [16] [17] [19] [20] [21] [28] [31] [32]. In particular, the HX preconditioner proposed in [17] is very popular. The action of the HX preconditioner is implemented by solving four Laplace subproblems, so the existing codes for Laplace equations can be easily used to solve (1.1). It is well known that the (orthogonal or regular) Helmholtz decomposition with stable estimates (see [14] [11]) plays an essential role in the convergence analysis of the kinds of preconditioners. For example, the HX preconditioner has been shown, by using the classic regular Helmholtz decomposition, to possess the optimal convergence for the case with constant coefficients. Although numerical results indicate that this preconditioner is still stable for some examples with large jump coefficients [23], it seems a theoretical open problem whether the results in [17] still hold for the case that the coefficients α and β have large jumps (refer to Subsection 7.3 of [17]). This topic was discussed in [35] for the case with two subdomains (i.e., the interface problems). The main difficulty is that the estimates in the classic Helmholtz decomposition are stable only with respect to the standard norms, which do not involve the coefficients α and β.

The first important attempt for Helmholtz decomposition in nonhomogeneous medium was made in [19], where a weighted discrete (orthogonal) Helmholtz decomposition, which is almost stable with respect to the weight function β, was constructed and studied. Moreover, in that paper the desired convergence result of the preconditioner proposed in [21] was proved by using this weighted Helmholtz decomposition. Unfortunately, the weighted discrete Helmholtz decomposition constructed in [19] cannot be applied to analyze the HX preconditioner for the case with large jump coefficients.

This paper is the second one of two serial articles. In the current paper, based on the auxiliary results derived in the first paper [18] and absorbing some ideas presented in [19], we build new discrete regular Helmholtz-type decompositions, which are nearly stable with respect to the mesh size h and are uniformly stable with respect to the weight norms involving the coefficients α and β, even if the coefficients α and β have large jumps across two neighboring media. We would like to emphasize two key differences between the results obtained in the paper and [19]: (1) both the jumps of the coefficients α and β are handled in this paper; (2) the results obtained in this paper covers all the cases of the distribution of the coefficient α, but some assumptions on the distribution of the coefficient β was imposed in [19]. By using this regular Helmholtz decomposition, we show that the PCG method with the HX preconditioner for solving the considered Maxwell system has a nearly optimal convergence rate, which grows only as the logarithm of the dimension of the underlying Nedelec finite element space, and more importantly, is independent of the jumps of the coefficients α and β across the interface between two neighboring subdomains.

The outline of the paper is as follows. In Section 2, we describe domain decomposition based on the distribution of coefficients, and define some edge finite element subspaces. In section 3, we describe two new regular Helmholtz decompositions and analyze the HX preconditioner for the case with strongly discontinuous coefficients by using the new regular Helmholtz decompositions. A new regular Helmholtz decomposition is constructed and analyzed in Section 4 for a particular case. The new regular Helmholtz decomposition for the general case is constructed and analyzed in Section 5.

2 Subdomains, finite element spaces

This section shall introduce subdomain decompositions and some fundamental finite element spaces.

2.1 Sobolev spaces

For an open and connected bounded domain $\mathcal{O}$ in $\mathbf{R}^3$, let $H_0^1(\mathcal{O})$ be the standard Sobolev space. Define the **curl**-spaces as follows

$$H(\mathbf{curl};\mathcal{O}) = \{\mathbf{v} \in L^2(\mathcal{O})^3;\ \mathbf{curl}\,\mathbf{v} \in L^2(\mathcal{O})^3\}$$

and

$$H_0(\mathbf{curl};\mathcal{O}) = \{\mathbf{v} \in H(\mathbf{curl};\mathcal{O});\ \mathbf{v}\times\mathbf{n} = 0 \text{ on } \partial\mathcal{O}\}.$$

2.2 Domain decomposition based on the distribution of coefficients

The main goal of this paper is to present a regular Helmholtz decomposition based on a decomposition of the global domain Ω into a set of non-overlapping subdomains so that the Helmholtz decomposition is nearly stable with respect to a discontinuous weight function related to the subdomains. For this purpose, we first decompose the entire domain Ω into subdomains based on the discontinuity of the weight function defined by the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ of (1.1) in applications.

Associated with the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ in (1.1), we assume that the entire domain Ω can be decomposed into N_0 open polyhedral subdomains $\Omega_1, \Omega_2, \cdots, \Omega_{N_0}$ such that $\bar{\Omega} = \cup_{k=1}^{N_0}\bar{\Omega}_k$ and the variations of the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ are not large in each subdomain Ω_k. Without loss of generality, we assume that, for $k = 1, 2, \ldots, N_0$,

$$\alpha(\mathbf{x}) = \alpha_r \quad\text{and}\quad \beta(\mathbf{x}) = \beta_r, \quad \forall\, \mathbf{x} \in \Omega_k, \tag{2.1}$$

where each α_k or β_k is a positive constant. Such a decomposition is possible in many applications when Ω is formed by multiple media. Notice that a subdomain Ω_k may be a non-convex polyhedron, which is a union of several convex polyhedra. In this sense our assumption is not restrictive and does cover many practical cases.

Remark 2.1 *The subdomains* $\{\Omega_k\}_{k=1}^{N_0}$ *are of different nature from those in the context of the standard domain decomposition methods:* $\{\Omega_k\}_{k=1}^{N_0}$ *is decomposed based only on the distribution of the jumps of the coefficient* $\alpha(x)$ *and* $\beta(x)$ *(so* N_0 *is a fixed integer, and the size of each* Ω_k *is* $O(1)$*).*

2.3 Edge and nodal element spaces

Next, we further divide each Ω_k into smaller tetrahedral elements of size h so that all the elements on Ω constitute a quasi-uniform triangulation $\mathcal{T}_h$ of the domain Ω. Let $\mathcal{E}_h$ and $\mathcal{N}_h$ denote the set of edges and nodes of $\mathcal{T}_h$ respectively. Then the Nedelec edge element space, of the lowest order, is a subspace of piecewise linear polynomials defined on $\mathcal{T}_h$:

$$V_h(\Omega) = \Big\{\mathbf{v} \in H_0(\mathbf{curl};\Omega);\ \mathbf{v}|_K \in R(K),\ \forall K \in \mathcal{T}_h\Big\},$$

where $R(K)$ is a subset of all linear polynomials on the element K of the form:

$$R(K) = \Big\{\mathbf{a} + \mathbf{b} \times \mathbf{x};\ \mathbf{a}, \mathbf{b} \in \mathbf{R}^3,\ \mathbf{x} \in K\Big\}.$$

It is known that for any $\mathbf{v} \in V_h(\Omega)$, its tangential components are continuous on all edges of each element in the triangulation $\mathcal{T}_h$, and $\mathbf{v}$ is uniquely determined by its moments on each edge e of $\mathcal{T}_h$:

$$M_h(\mathbf{v}) = \Big\{\lambda_e(\mathbf{v}) = \int_e \mathbf{v} \cdot \mathbf{t}_e ds;\ e \in \mathcal{E}_h\Big\}$$

where $\mathbf{t}_e$ denotes the unit vector on edge e, and this notation will be used to denote any edge or union of edges, either from an element $K \in \mathcal{T}_h$ or from a subdomain. For a vector-valued function $\mathbf{v}$ with appropriate smoothness, we introduce its edge element interpolation $\mathbf{r}_h\mathbf{v}$ such that $\mathbf{r}_h\mathbf{v} \in V_h(\Omega)$, and $\mathbf{r}_h\mathbf{v}$ and $\mathbf{v}$ have the same moments as in $M_h(\mathbf{v})$. The interpolation operator $\mathbf{r}_h$ will be used in the construction of a stable decomposition for any function $\mathbf{v}_h \in V_h(\Omega)$.

As we will see, the edge element analysis involves also frequently the nodal element space. For this purpose we introduce $Z_h(\Omega)$ to be the standard continuous piecewise linear finite element space in $H_0^1(\Omega)$ associated with the triangulation $\mathcal{T}_h$.

2.4 Finite element subspaces

For the subsequent analysis, we need the subspaces of the global edge element space $V_h(\Omega)$ restricted on a subdomain of Ω.

Let G be any of the subdomains $\Omega_1, \cdots, \Omega_{N_0}$ of Ω. We will often use F, E and V to denote a general face, edge and vertex of G respectively, but use e to denote a general edge of $\mathcal{T}_h$ lying on ∂G. Associated with G, we write the natural restriction of $V_h(\Omega)$ and $Z_h(\Omega)$ on G by $V_h(G)$ and $Z_h(G)$, respectively. Define

$$V_h(\partial G) = \{(\mathbf{v} \times \mathbf{n})|_{\partial G};\ \mathbf{v} \in V_h(G)\},$$

$$Z_h^0(G) = \{q \in Z_h(G);\ q = 0 \text{ on } \partial G\}$$

and

$$V_h^0(G) = \{\mathbf{v} \in V_h(G);\ \mathbf{v} \times \mathbf{n} = \mathbf{0} \text{ on } \partial G\}.$$

3 Discrete Helmholtz decompositions and HX preconditioner with jump coefficients

In this section, we describe new regular Helmholtz decompositions, which are stable uniformly with the weight functions.

3.1 Motives

As is well known, the (orthogonal or regular) Helmholtz decomposition plays an essential role in the convergence analysis of the multigrid and non-overlapping domain decomposition methods for solving the Maxwell system (1.1) by edge element methods; see, e.g., [2] [16] [17] [20] [21] [28] [29] [31] [32]. Any vector valued function $\mathbf{v} \in H_0(\mathbf{curl};\Omega)$ admits a regular Helmholtz decomposition of the form (see, for example, [11] and [28])

$$\mathbf{v} = \nabla p + \mathbf{w} \tag{3.1}$$

for some $p \in H_0(\Omega)$ and $\mathbf{w} \in (H_0(\Omega))^3$, and have the following stability estimates

$$\|p\|_{1,\Omega} \le C\|\mathbf{v}\|_{\mathbf{curl},\Omega}\,, \quad \|\mathbf{w}\|_{1,\Omega} \le C\|\mathbf{curl}\,\mathbf{v}\|_{0,\Omega}\,. \tag{3.2}$$

In order to effectively deal with the case with jump coefficients in (1.1), one hopes the stability estimates (3.2) to be still held with the weighted norms defined by the weight function α (*or* β). Unfortunately, it is unclear how the coefficient C appearing in the two stability estimates depends on the jumps of the coefficients α and β across the interface between two neighboring subdomains. For this reason, although there are many preconditioners available in the literature for the Maxwell system (1.1), with optimal or nearly optimal convergence in terms of the mesh size, it is still unclear how the convergence depend on the jumps of the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ in (1.1). For example, the well known HX preconditioner proposed in [17] has been shown to possess the optimal convergence for the case with continuous coefficients, but it seems a theoretical open problem whether the result still hold for the case with large jump coefficients (refer to Subsection 7.3 of [17]). The key tool used in [17] is a discrete regular Helmholtz decomposition derived by (3.1) and (3.2).

The aim of this work is to fill in this gap by constructing new discrete regular Helmholtz-type decompositions, that are stable uniformly with respect to the jumps of the weight coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$. The new regular Helmholtz decompositions can be used to analyze convergence of various preconditioners for Maxwell's equations with large jumps in coefficients. For an application, we will show in Section 3.4 with the help of such a Helmholtz decomposition that the HX preconditioner constructed in [17] converges not only nearly optimally in terms of the finite element mesh size, but also independently of the jumps in the coefficients $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ in (1.1).

From now on, we shall frequently use the notations $\lesssim$ and $\eqsim$. For any two non-negative quantities x and y, $x \lesssim y$ means that $x \le Cy$ for some constant C independent of mesh size h, subdomain size d and the possible large jumps of some related coefficient functions across the interface between any two subdomains. $x \eqsim y$ means $x \lesssim y$ and $y \lesssim x$.

3.2 Discrete Helmholtz decomposition under the quasi-monotonicity assumption

In the analysis of multilevel preconditioner for the case with large jump coefficient, the *quasi-monotonicity assumption* was usually made in the existing literature (see, for example, [13] and [26]).

3.2.1 The quasi-monotonicity assumption

We first recall the definition of quasi-monotonicity assumption (see **Defintion 4.1** and **Defintion 4.6** in [26]).

Let $\mathcal{N}(\Omega)$ (rep. $\mathcal{N}(\partial\Omega)$) denote the set of vertices V in Ω (rep. on $\partial\Omega$). In the following, we always use V to denote a vertex from the domain decomposition, namely, V is a vertex of some polyhedron Ω_r. For a vertex V, let Ξ_{V} denote the union of all polyhedra Ω_r that contain V as one of their vertices. Denote by $\tilde{\Omega}_{\mathrm{V}}$ a polyhedron from Ξ_{V} such that the maximum $\max_{\mathbf{x}\in\Xi_{\mathrm{V}}}\alpha(\mathbf{x})$ achieves on $\tilde{\Omega}_{\mathrm{V}}$.

Definition 3.1. The distribution of the coefficients $\{\alpha_k\}$ satisfying $\Omega_k\subset\Xi_{\mathrm{V}}$ will be called *quasi-monotone* with respect to the vertex V if the following conditions are fulfilled: For each $\Omega_r\subset\Xi_{\mathrm{V}}$ there exists a Lipschitz domain $\tilde{\Xi}_{\mathrm{V},r}$ containing only polyhedra from Ξ_{V}, such that

• if $\mathrm{V}\in\mathcal{N}(\Omega)$ then $\Omega_r\cup\tilde{\Omega}_{\mathrm{V}}\subseteq\tilde{\Xi}_{\mathrm{V},r}$ and $\alpha_r\leq\alpha_{r'}$ for any $\Omega_{r'}\subseteq\tilde{\Xi}_{\mathrm{V},r}$;

• if $\mathrm{V}\in\mathcal{N}(\partial\Omega)$ then $\Omega_r\subseteq\tilde{\Xi}_{\mathrm{V},r}$, $meas(\partial\tilde{\Xi}_{\mathrm{V},r}\cap\partial\Omega)>0$ (namely, $\partial\tilde{\Xi}_{\mathrm{V},r}\cap\partial\Omega$ is just a face of some polyhedron Ω_k) and $\alpha_r\leq\alpha_{r'}$ for any $\Omega_{r'}\subseteq\tilde{\Xi}_{\mathrm{V},r}$.

The distribution of the coefficients $\alpha_1,\alpha_2,\cdots,\alpha_{N_0}$ is *quasi-monotone* with respect to vertices generated by the domain decomposition, if the above conditions hold for every vertices V. Similarly, we can define the *quasi-monotonicity* of the coefficients $\alpha_1,\alpha_2,\cdots,\alpha_{N_0}$ with respect to edges E generated by the domain decomposition.

We say that the distribution of the coefficients $\alpha_1,\alpha_2,\cdots,\alpha_{N_0}$ satisfies *quasi-monotonicity assumption* if it is *quasi-monotone* with respect to both vertices and edges generated by the domain decomposition.

3.2.2 The main result

The stability estimates are based on some weighted norms. For $H(\mathbf{curl})$ functions, we define

$$\|\mathbf{v}\|_{L^2_\alpha(\Omega)}=(\sum_{r=1}^{N_0}\alpha_r\|\mathbf{v}\|^2_{0,\Omega_r})^{\frac{1}{2}},\quad \mathbf{v}\in H(\mathbf{curl};\ \Omega)$$

and

$$\|\mathbf{v}\|_{H^*(\mathbf{curl},\Omega)}=(\sum_{r=1}^{N_0}\alpha_r\|\mathbf{curl}\ \mathbf{v}\|^2_{0,\Omega_r}+\beta_r\|\mathbf{v}\|^2_{0,\Omega_r})^{\frac{1}{2}},\quad \mathbf{v}\in H(\mathbf{curl};\ \Omega).$$

For H^1 functions, we define

$$\|p\|_{H^1_\beta(\Omega)}=(\sum_{r=1}^{N_0}\beta_r\|\nabla p\|^2_{0,\Omega_r}+\beta_r\|p\|^2_{0,\Omega_r})^{\frac{1}{2}},\quad p\in H^1(\Omega)$$

and

$$\|\mathbf{v}\|_{H^1_*(\Omega)}=(\sum_{r=1}^{N_0}(\alpha_r|\mathbf{v}|^2_{1,\Omega_r}+\beta_r\|\mathbf{v}\|^2_{0,\Omega_r}))^{\frac{1}{2}},\quad \mathbf{v}\in(H^1(\Omega))^3.$$

The stable Helmholtz decomposition involves an assumption on the coefficients α and β.

Assumption 3.1. There is a constant C such that, for any two neighboring subdomains Ω_i and Ω_j, we have

$$\beta_i\leq C\beta_j\quad\text{when}\quad\alpha_i\leq\alpha_j. \tag{3.3}$$

For the case considered in this subsection, we have the following result

Theorem 3.1 *Assume that* **Assumption 3.1** *is satisfied. Then, under the quasi-monotonicity assumption, any function* $\mathbf{v}_h \in V_h(\Omega)$ *admits a decomposition of the form*

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h \tag{3.4}$$

for some $p_h \in Z_h(\Omega)$ *and* $\mathbf{w}_h \in (Z_h(\Omega))^3$ *and* $\mathbf{R}_h \in V_h(\Omega)$*. Moreover,* p_h*,* $\mathbf{w}_h$ *and* $\mathbf{R}_h$ *have the estimates*

$$\|p_h\|_{H^1_\beta(\Omega)} \leq C \log^m(1/h) \|\beta^{\frac{1}{2}} \mathbf{v}_h\|_{0,\Omega}, \tag{3.5}$$

$$\|\mathbf{w}_h\|_{H^1_*(\Omega)} \leq C \log^m(1/h) \|\mathbf{v}_h\|_{H^*(\mathbf{curl},\ \Omega)} \tag{3.6}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{L^2_\alpha(\Omega)} \leq C \log^m(1/h) \|\mathbf{curl}\, \mathbf{v}_h\|_{L^2_\alpha(\Omega)}, \tag{3.7}$$

where constants m (≥ 2) *and* C *are independent of* h *and the jumps of the coefficients* α *and* β*.*

We consider the example tested in the paper [23] to illustrate the assumptions in this theorem.

Let $\Omega = [0,1]^3$ be the unit cube, and be divided into two polyhedrons D_1 and D_2 (see Figure 1). There are two choices of the coefficients: (a) $\alpha \equiv 1$ on Ω, $\beta = 1$ on D_1 and $\beta = 10^k$ (with $k = -8, \cdots, 8$) on D_2; (b) $\beta \equiv 1$ on Ω, $\alpha = 1$ on D_1 and $\alpha = 10^k$ (with $k = -8, \cdots, 8$) on D_2. A uniform triangulation is used on Ω.

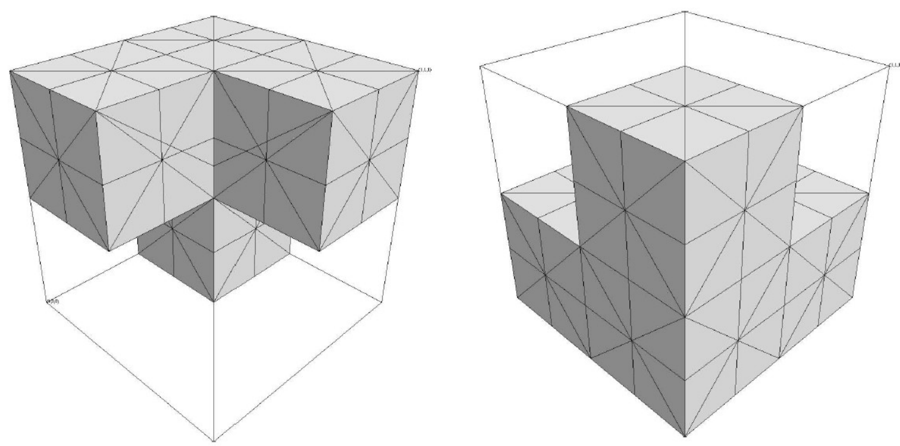

Figure 1: The unit cube split into two symmetrical regions (left: D_1; right: D_2)

For this example, there is no internal cross point and so the quasi-monotonicity assumption is satisfied. For Case (a), we have $\alpha_1 = \alpha_2$ and $\beta_1 = \alpha_2 < \beta_2$ when $k = 1, \cdots, 8$; we have $\alpha_2 = \alpha_1$ and $\beta_2 < \beta_1 = \alpha_1$ when $k = -8, \cdots, 0$, so **Assumption 3.1** is met. For Case (b), when $k = -8, \cdots, 0$ we have $\alpha_2 \leq \alpha_1$ and $\beta_2 = \alpha_1 = \beta_1$; when $k = 1, \cdots, 8$ we have $\alpha_1 < \alpha_2$ and $\beta_1 = \alpha_2 = \beta_2$, so **Assumption 3.1** is also satisfied.

3.3 Discrete Helmholtz decomposition for the general case

In this subsection, we consider the complicated case that the quasi-monotonicity assumption does not hold.

3.3.1 A new concept

For convenience, we give another concept. For a vertex $\mathrm{V} \in \mathcal{N}(\Omega) \cup \mathcal{N}(\partial\Omega)$, let $\Im_{\mathrm{V}}$ denote the set of all polyhedra Ω_r that contain V as one of their vertices.

Definition 3.2. A vertex V is called a *weird vertex* if there is some polyhedron $\Omega_r \in \Im_{\mathrm{V}}$ such that one of the following two conditions is satisfied: (i) for any other polyhedron $\Omega_{r'}$ that belongs to $\Im_{\mathrm{V}}$ and corresponds to larger coefficient than Ω_r (i.e., $\alpha_{r'} \geq \alpha_r$), the intersection of Ω_r with $\Omega_{r'}$ is just the vertex V, i.e., $\bar{\Omega}_r \cap \bar{\Omega}_{r'} = \mathrm{V}$; (ii) $\bar{\Omega}_r \cap \partial\Omega = \mathrm{V}$ (thus $\mathrm{V} \in \mathcal{N}(\partial\Omega)$) and the local maximum $\max_{\mathbf{x}\in\Xi_{\mathrm{V}}} \alpha(\mathbf{x})$ achieves on $\bar{\Omega}_r$.

We would like to give the relations between *quasi-monotonicity assumption* and *weird vertex.*

Proposition 3.1. The distribution of the coefficients $\alpha_1, \alpha_2, \cdots, \alpha_{N_0}$ is *quasi-monotone* with respect to all the vertices implies that there is no *weird vertex* in $\mathcal{N}(\Omega) \cup \mathcal{N}(\partial\Omega)$. But, the inverse conclusion is not valid.

Proof. If the distribution of the coefficients is *quasi-monotone* with respect to a vertex V, then the set $\tilde{\Xi}_{\mathrm{V},r}$ in **Definition 3.1** must be a Lipschitz polyhedron (may be non-convex). This condition implies that the set $(\tilde{\Xi}_{\mathrm{V},r} \backslash \Omega_r) \cap \bar{\Omega}_r$ is just a face of Ω_r (otherwise, $\tilde{\Xi}_{\mathrm{V},r}$ is a non-Lipschitz domain), so the vertex V is not a weird point (notice that $\alpha_{r'} \geq \alpha_r$ for any $\Omega_{r'} \subset \tilde{\Xi}_{\mathrm{V},r} \backslash \Omega_r$). But, the vertex V is not a weird vertex means that the set $(\tilde{\Xi}_{\mathrm{V},r} \backslash \Omega_r) \cap \bar{\Omega}_r$ is a face or an edge of Ω_r, from which we can not infer quasi-monotonicity of the coefficients with respect to the vertex V. ♯

Remark 3.1 *For a weird vertex* V*, the set* $\Im_{\mathrm{V}}$ *can be decomposed into two disjoint sets* $\Im^*_{\mathrm{V}}$ *and* $\Im^c_{\mathrm{V}}$ *that can be described as follows: for any polyhedron* $\Omega_r \in \Im^*_{\mathrm{V}}$ *there are at least one polyhedron* $\Omega_{r'} \in \Im_{\mathrm{V}}$ *such that* $\alpha_{r'} \geq \alpha_r$ *and the intersection of* Ω_r *and* $\Omega_{r'}$ *is a face or an edge containing* V*; for any two polyhedrons* Ω_r, $\Omega_l \in \Im^c_{\mathrm{V}}$*, we have* $\bar{\Omega}_r \cap \bar{\Omega}_l = \mathrm{V}$*. It is easy to see that both* $\Im^*_{\mathrm{V}}$ *and* $\Im^c_{\mathrm{V}}$ *are not the empty set (*$\Im^c_{\mathrm{V}}$ *at least contains the polyhedron* Ω_r *mentioned in* **Definition 3.1** *and the polyhedron achieving the locally maximal coefficient). If* V *is not a weird vertex, then* $\Im^c_{\mathrm{V}} = \emptyset$ *and* $\Im^*_{\mathrm{V}} = \Im_{\mathrm{V}}$*.*

As we will see, the introduction of weird vertices can help us to analyze convergence of the HX preconditioner for more complicated situations.

3.3.2 The main result

In this part we describe a Helmholtz decomposition for the case with *weird vertices*.

Let $\mathcal{V}_s$ denote the set of all the weird vertices generated by the domain decomposition and the distribution of the coefficients. For $\mathrm{V} \in \mathcal{V}_s$, we use n_{V} to denote the number of the subdomains contained in $\Im^c_{\mathrm{V}}$. Let $\mathcal{V}^{in}_s$ (resp. $\mathcal{V}^b_s$) denote the set of the weird vertices in Ω (resp. on $\partial\Omega$). Define

$$n_s = \sum_{\mathrm{V}\in\mathcal{V}^{in}_s} (n_{\mathrm{V}} - 1) + \sum_{\mathrm{V}\in\mathcal{V}^b_s} n_{\mathrm{V}}.$$

For convenience, the number n_s is called *multiplicity of weird vertices*, which reflects the number of weird vertices and the distribution of the coefficients on the polyhedron subdomains containing a weird vertex as one of their vertices.

For n_s functionals $\{\mathcal{F}_l\}_{l=1}^{n_s}$, each of which corresponds to a weird vertex, we define

$$V^*_h(\Omega) = \{\mathbf{v}_h \in V_h(\Omega) : \ \mathcal{F}_l \mathbf{v}_h = 0 \ \text{ for } \ l = 1, \cdots, n_s\}.$$

Of course, when there is no *weird vertex*, we have $V^*_h(\Omega) = V_h(\Omega)$. It is clear that $\dim(V^*_h(\Omega)) = \dim(V_h(\Omega)) - n_s$, so the number n_s is the codimension of the space $V^*_h(\Omega)$. The exact definitions of the functionals $\{\mathcal{F}_l\}_{l=1}^{n_s}$ will be given in Section 5.

In applications, it is particularly difficult to efficiently solve Maxwell's equations for the case with small coefficient $\beta(\mathbf{x})$. Thus we give a usual assumption below
Assumption 3.2. The coefficient functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$ satisfy

$$\beta_r/\alpha_r \leq C \quad (r = 1, \cdots, N_0). \tag{3.8}$$

The following theorem presents another main result of this paper.

Theorem 3.2 *Assume that both* **Assumption 3.1** *and* **Assumption 3.2** *are satisfied. Then there are functionals* $\{\mathcal{F}_l\}_{l=1}^{n_s}$ *such that any function* $\mathbf{v}_h \in V_h^*(\Omega)$ *admits a decomposition of the form*

$$\mathbf{v}_h = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h \tag{3.9}$$

for some $p_h \in Z_h(\Omega)$ *and* $\mathbf{w}_h \in (Z_h(\Omega))^3$ *and* $\mathbf{R}_h \in V_h(\Omega)$*. Moreover, we have the estimates*

$$\|\beta^{\frac{1}{2}} \nabla p_h\|_{0,\Omega} \leq C \log^{\hat{m}}(1/h) \|\mathbf{v}_h\|_{H^*(\mathbf{curl},\ \Omega)}, \tag{3.10}$$

$$\|\mathbf{w}_h\|_{H_*^1(\Omega)} \leq C \log^{\hat{m}}(1/h) \|\mathbf{curl}\, \mathbf{v}_h\|_{L_\alpha^2(\Omega)} \tag{3.11}$$

and

$$h^{-1} \|\mathbf{R}_h\|_{L_\alpha^2(\Omega)} \leq C \log^{\hat{m}}(1/h) \|\mathbf{curl}\, \mathbf{v}_h\|_{L_\alpha^2(\Omega)}, \tag{3.12}$$

where constants $\hat{m}$ *and* C *are independent of* h *and the jumps of the coefficients* α *and* β*. If there is no weird vertex (this condition is weaker than the quasi-monotonicity assumption in Theorem 3.2), the results are valid for any* $\mathbf{v}_h \in V_h(\Omega)$*.*

As we will see, when we apply the above theorem to the analysis of the HX preconditioner, we are interested only in the codimension n_s of the space $V_h^*(\Omega)$, instead of the space $V_h^*(\Omega)$ itself (i.e., the choice of the functionals $\{\mathcal{F}_l\}_{l=1}^{n_s}$). In many applications, one may encounter only several different media involved in the entire physical domain, so n_s is a small positive integer independent of h and the jumps of the coefficients α and β.

3.3.3 Further investigation on this theorem

To understand Theorem 3.1 more deeply, we give a well known example on the so called "checkerboard" domain.

Let $\Omega = [0,1]^3$, and set $\Omega_1 = [0, \frac{1}{2}]^3 \cup [\frac{1}{2}, 1]^3$ and $\Omega_2 = \Omega \backslash \Omega_1$. Define $\alpha(\mathbf{x}) = \beta(\mathbf{x}) = 1$ on Ω_1, and $\alpha(\mathbf{x}) = \beta(\mathbf{x}) = \varepsilon$ on Ω_2 with $\varepsilon \ll 1$. It is easy to see that the coefficients in this example satisfy **Assumption 3.1** and **Assumption 3.2**. The domain $\Omega = [0,1]^3$ in the example is called "checkerboard" domain. For this example, there is only one weird vertex v at the center of Ω and $\Im_{\mathrm{V}}^c$ contains two cubes, which implies that $n_s = 1$.
Proposition 3.2. For the "checkerboard" domain, the space $V_h^*(\Omega)$ in Theorem 3.2 cannot be replaced by $V_h(\Omega)$ itself.
Proof. For convenience, set $G_1 = [0, \frac{1}{2}]^3$ and $G_2 = [\frac{1}{2}, 1]^3$, and let v be the common vertex of G_1 and G_2 (see Figure 2).

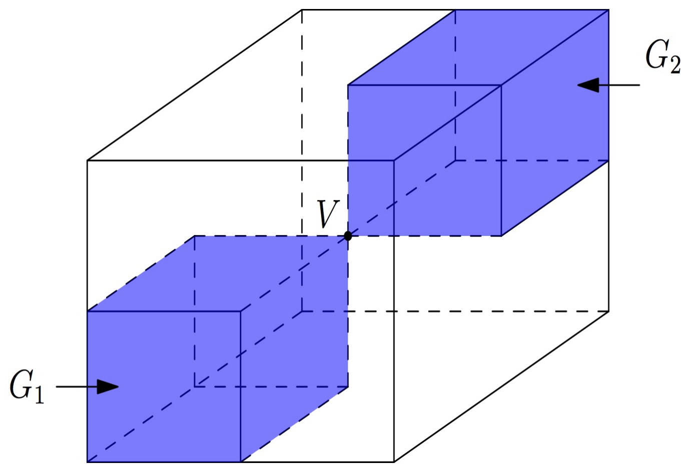

Figure 2: A "checkerboard" domain Ω: the shaded domain denotes Ω_1

For $i = 1, 2$, let $\phi_{h,i} \in Z_h(G_i)$ be a nodal finite element function satisfying $\phi_{h,i} = 0$ on $\partial G_i \cap \partial\Omega$. Define $\mathbf{v}_h$ as follows: $\mathbf{v}_h = \nabla\phi_{h,i}$ on G_i; $\lambda_e(\mathbf{v}_h) = 0$ for any edge e in Ω_2. It is easy to see that $\mathbf{v}_h \in V_h(\Omega)$ even if $\phi_{h,1} \neq \phi_{h,2}$ at V (an edge finite element function may be discontinuous at a node), but $\mathbf{v}_h$ does not vanish on Ω_2 since $\mathbf{v}_h$ has non-zero degrees of freedom on $\partial\Omega_2 \cap \partial\Omega_1$. In the following we explain that the function $\mathbf{v}_h$ must belong to a subspace $V_h^*(\Omega)$ with some constrain if this function admits a Helmholtz decomposition satisfying all the requirements in Theorem 3.2.

We assume that $\mathbf{v}_h$ admits such a Helmholtz decomposition. By the definition of $\mathbf{v}_h$, we have $\mathbf{curl}\,\mathbf{v}_h = \mathbf{0}$ on G_i $(i = 1, 2)$. Then the estimates (3.11) and (3.12) imply that

$$\|\mathbf{w}_h\|_{0,\Omega_1} \leq \|\mathbf{w}_h\|_{H^1_*(\Omega)} \leq C\log^{m+1}(1/h)\varepsilon^{\frac{1}{2}}\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_2}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{0,\Omega_1} \leq h^{-1}\|\mathbf{R}_h\|_{L^2_\alpha(\Omega)} \leq C\log^{m+1}(1/h)\varepsilon^{\frac{1}{2}}\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_2}.$$

For a fixed h, let $\varepsilon \to 0^+$. Then, from the above inequalities, we get $\mathbf{w}_h = \mathbf{R}_h = \mathbf{0}$ on Ω_1. Thus, by the Helmholtz decomposition, we have $\mathbf{v}_h = \nabla p_h$ on Ω_1 with $p_h \in Z_h(\Omega)$.

For $i = 1, 2$, set $p_{h,i} = p_h|_{G_i}$ and let $\mathrm{F}_i \subset \partial G_i$ be a face containing V as one of its vertex. For this example, we have $\mathrm{F}_i \cap \partial\Omega \neq \emptyset$, so we can choose a vertex V_i on $\partial\mathrm{F}_i \cap \partial\Omega$ such that $p_{h,i}$ vanishes at V_i. We want to consider the arc-length integral on $\partial\mathrm{F}_i$. To this end, we assume that the arc-length coordinate of the point V_i is just 0 and we use t_V to denote the arc-length coordinate of the point V. By the condition $\mathbf{v}_h = \nabla p_h$ on G_i, we get

$$\int_0^{t_\mathrm{V}} \mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}_i} ds = \int_0^{t_\mathrm{V}} \nabla p_{h,i} \cdot \mathbf{t}_{\partial\mathrm{F}_i} ds = p_{h,i}(t_\mathrm{V}) \quad (i = 1, 2).$$

Since $p_h \in Z_h(\Omega)$, we have $p_{h,1}(t_\mathrm{V}) = p_{h,2}(t_\mathrm{V})$. Thus

$$\int_0^{t_\mathrm{V}} \mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}_1} ds = \int_0^{t_\mathrm{V}} \mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}_2} ds.$$

Namely, the function $\mathbf{v}_h$ must satisfy the constraint $\mathcal{F}\mathbf{v}_h = 0$ with

$$\mathcal{F}\mathbf{v}_h = \int_0^{t_\mathrm{V}} \mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}_1} ds - \int_0^{t_\mathrm{V}} \mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}_2} ds.$$

Then this proposition is proved. $\sharp$

The above discussions tell us that, for the case with weird vertices, some constraint is necessary for a function $\mathbf{v}_h$ to admit a stable Helmholtz decomposition.

3.4 Analysis of the HX preconditioner with jump coefficients

In this subsection we shall apply the discrete weighted Helmholtz decompositions described in Theorem 3.1-Theorem 3.2 to analyze the convergence of the HX preconditioner for the case with jump coefficients.

3.4.1 The HX preconditioner

In this part, we recall the HX preconditioner proposed in [17] for solving the discrete system of (1.1).

The discrete variational problem of (1.1) is: to find $\mathbf{u}_h \in V_h(\Omega)$ such that

$$(\alpha \mathbf{curl}\, \mathbf{u}_h, \mathbf{curl}\, \mathbf{v}_h) + (\beta \mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in V_h(\Omega). \tag{3.13}$$

As usual, we can rewrite it in the operator form

$$\mathbf{A}_h \mathbf{u}_h = \mathbf{f}_h, \tag{3.14}$$

with $\mathbf{A}_h : V_h(\Omega) \rightarrow V_h(\Omega)$ being defined by

$$(\mathbf{A}_h \mathbf{u}_h, \mathbf{v}_h) = (\alpha \mathbf{curl}\, \mathbf{u}_h, \mathbf{curl}\, \mathbf{v}_h) + (\beta \mathbf{u}_h, \mathbf{v}_h), \quad \mathbf{u}_h, \mathbf{v}_h \in V_h(\Omega).$$

Let $\triangle_h : (Z_h(\Omega))^3 \rightarrow (Z_h(\Omega))^3$ be the discrete elliptic operator defined by

$$(\triangle_h \mathbf{v}, \mathbf{w}) = (\alpha \nabla \mathbf{v}, \nabla \mathbf{w}) + (\beta \mathbf{v}, \mathbf{w}), \quad \mathbf{v}, \mathbf{w} \in (Z_h(\Omega))^3,$$

and let $\vartriangle_h$: $\nabla(Z_h(\Omega)) \rightarrow \nabla(Z_h(\Omega))$ be the restriction of $\mathbf{A}_h$ on the space $\nabla(Z_h(\Omega))$, whose action can be implemented by solving Laplace equation. Besides, let $\mathbf{J}_h : V_h(\Omega) \rightarrow V_h(\Omega)$ denote the Jacobi smoother of $\mathbf{A}_h$. Then the HX preconditioner $\mathbf{B}_h$ of $\mathbf{A}_h$ can be defined by

$$\mathbf{B}_h = \mathbf{J}_h^{-1} + \hat{\mathbf{r}}_h \triangle_h^{-1} \hat{\mathbf{r}}_h^* + \mathbf{T}_h \vartriangle_h^{-1} \mathbf{T}_h^*,$$

where $\hat{\mathbf{r}}_h : (Z_h(\Omega))^3 \rightarrow V_h(\Omega)$ is the restriction of the interpolation operator $\mathbf{r}_h$ on $(Z_h(\Omega))^3$, and $\mathbf{T}_h^* : V_h(\Omega) \rightarrow \nabla(Z_h(\Omega))$ is the L^2 projector.

When the coefficients α and β have no large jump across the interface between two neighboring subdomains, we have (see [17])

$$cond(\mathbf{B}_h \mathbf{A}_h) \lesssim C.$$

But, it is unclear how the constant C depends on the jumps of the coefficients α and β for the case with large jumps of the coefficients.

3.4.2 Convergence of the HX preconditioner for the case with jump coefficients

In this subsection we give a new convergence result of the preconditioner $\mathbf{B}_h$ for the case with large jumps of the coefficients α and β by using Theorem 3.1-Theorem 3.2.

Let n_s and $V_h^*(\Omega)$ be defined in Subsubsection 3.3.2. We use $\lambda_{n_s+1}(\mathbf{B}_h^{-1}\mathbf{A}_h)$ to denote the minimal eigenvalue of the restriction of $\mathbf{B}_h^{-1}\mathbf{A}_h$ on the subspace $V_h^*(\Omega)$, and define $\kappa_{n_s+1}(\mathbf{B}_h^{-1}\mathbf{A}_h)$ as the *reduced condition number* (see [34]) of $\mathbf{B}_h^{-1}\mathbf{A}_h$ associated with the subspace $V_h^*(\Omega)$. Namely,

$$\kappa_{n_s+1}(\mathbf{B}_h^{-1}\mathbf{A}_h) = \frac{\lambda_{\max}(\mathbf{B}_h^{-1}\mathbf{A}_h)}{\lambda_{n_s+1}(\mathbf{B}_h^{-1}\mathbf{A}_h)}.$$

From the framework introduced in [34], we know that the convergence rate of the PCG method with the preconditioner $\mathbf{B}_h$ for solving the system (3.14) is determined by the *reduced condition number* $\kappa_{n_s+1}(\mathbf{B}_h^{-1}\mathbf{A}_h)$ (the iteration counts to achieve a given accuracy of the approximation weakly depends on the values of the codimension n_s). If there is no weird vertex (this condition is weaker than the quasi-monotonicity assumption), we have $n_s = 0$ and $V_h^*(\Omega) = V_h(\Omega)$, and so $\kappa_{n_s+1}(\mathbf{B}_h^{-1}\mathbf{A}_h)$ is just the standard condition number $cond(\mathbf{B}_h^{-1}\mathbf{A}_h)$. In this part, we are devoted to the estimate of $\kappa_{n_s+1}(\mathbf{B}_h^{-1}\mathbf{A}_h)$.

Theorem 3.3 *Assume that both* **Assumption 3.1** *and* **Assumption 3.2** *are satisfied. Then there are a positive number C and a positive integer m_0, which are independent of h and the jumps of the coefficients α and β, and only depend on the distribution of the discontinuity of the coefficient functions $\alpha(\mathbf{x})$ and $\beta(\mathbf{x})$, such that*

$$\kappa_{n_s+1}(\mathbf{B}_h^{-1}\mathbf{A}_h) \lesssim C\log^{m_0}(1/h). \tag{3.15}$$

When both **Assumption 3.1** *and the quasi-monotonicity assumption are satisfied, we have*

$$cond(\mathbf{B}_h^{-1}\mathbf{A}_h) \lesssim C\log^{m_0}(1/h). \tag{3.16}$$

Proof. We need only to consider the estimate (3.15), and we can prove another result in the same manner (but using Theorem 3.1). For any $\mathbf{v}_h \in V_h^*(\Omega)$, let $\mathbf{w}_h \in (Z_h(\Omega))^3$, $p_h \in Z_h(\Omega)$ and $\mathbf{R}_h \in V_h(\Omega)$ be defined by the decomposition in Theorem 3.2

$$\mathbf{v}_h = \mathbf{r}_h\mathbf{w}_h + \nabla p_h + \mathbf{R}_h. \tag{3.17}$$

By the auxiliary space technique for the construction of preconditioner (refer to [17]), we need only to verify

$$(\mathbf{J}\mathbf{R}_h, \mathbf{R}_h) + (\triangle_h\mathbf{w}_h, \mathbf{w}_h) + (\Delta_h\ (\nabla p_h), \nabla p_h) \leq C\log^{m_0}(1/h)(\mathbf{A}_h\mathbf{v}_h, \mathbf{v}_h). \tag{3.18}$$

It follows by (3.11) that

$$\begin{aligned}(\triangle_h\mathbf{w}_h, \mathbf{w}_h) &= \|\mathbf{w}_h\|^2_{H^1_*(\Omega)} \lesssim \log^{m_0}(1/h)\|\mathbf{v}_h\|^2_{H^*(\mathbf{curl};\ \Omega)} \\ &= \log^{m_0}(1/h)(\mathbf{A}_h\mathbf{v}_h, \mathbf{v}_h).\end{aligned} \tag{3.19}$$

From (3.10), we have

$$\begin{aligned}(\Delta_h\ (\nabla p_h), \nabla p_h) &\lesssim \|\beta^{\frac{1}{2}}\nabla p_h\|^2_{0,\Omega} \lesssim \log^{m_0}(1/h)(\|\alpha^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega} + \|\beta^{\frac{1}{2}}\mathbf{v}_h\|^2_{0,\Omega}) \\ &= \log^{m_0}(1/h)(\mathbf{A}_h\mathbf{v}_h, \mathbf{v}_h).\end{aligned} \tag{3.20}$$

Finally, we get by (3.12)

$$\begin{aligned}(\mathbf{J}\tilde{\mathbf{v}}_h, \tilde{\mathbf{v}}_h) &\lesssim h^{-2}\|\alpha^{\frac{1}{2}}\tilde{\mathbf{v}}_h\|^2_{0,\Omega} \lesssim \log^{m_0}(1/h)\|\alpha^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega} \\ &\leq \log^{m_0}(1/h)(\mathbf{A}_h\mathbf{v}_h, \mathbf{v}_h).\end{aligned} \tag{3.21}$$

Then (3.18) is a direct consequence of (3.19)-(3.21). ♯

Remark 3.2 *Notice that the codimension n_s is a small positive constant in applications. By Theorem 3.3 and the framework introduced in [34], the PCG method with the preconditioner $\mathbf{B}_h$ for solving the system (3.14) possesses fast convergence, which not only is nearly optimal with respect to the mesh size h but also is independent of the jumps in the coefficients α and β across the interface between two neighboring subdomains.*

The remaining part of this work is devoted to the proof of Theorem 3.1-Theorem 3.2. As we will see, the key idea is to divide all the subdomains $\{\Omega_r\}$ into groups according to the values of the coefficients $\{\alpha_r\}$.

4 Analysis for the case satisfying the quasi-monotonicity assumption

In this section, we are devoted to the proof of Theorem 3.1. For the analysis, we introduce a subset of the boundary of each subdomain. For a subdomain Ω_r $(1 \leq r \leq N_0)$, let Γ_r be a union of the (closed) intersection sets of $\bar{\Omega}_r$ with $\partial\Omega$ or $\bar{\Omega}_l$ $(l = 1, \cdots, N_0)$ that satisfying $\alpha_l \geq \alpha_r$. Namely,

$$\Gamma_r = \bigcup_{\alpha_l \geq \alpha_r} (\bar{\Omega}_r \cap \bar{\Omega}_l) \bigcup (\partial\Omega_r \cap \partial\Omega).$$

The subset Γ_r possesses the following property, which reveals the essence of the quasi-monotonicity assumption

Proposition 4.1. Under the quasi-monotonicity assumption, for each Ω_r the set Γ_r is just a union of some faces of Ω_r.

Proof. If Γ_r contains an isolated vertex V, then there is a polyhedron $\Omega_{r'}$ with $\alpha_{r'} \geq \alpha_r$ such that $\mathrm{V} = \bar{\Omega}_r \cap \bar{\Omega}_{r'}$, and all the polyhedrons Ω_j having a common face or a common edge with Ω_r possesses the property $\alpha_j \leq \alpha_r$ (otherwise, $\bar{\Omega}_r \cap \bar{\Omega}_j \subset \Gamma_r$ and so $\mathrm{V} \in \bar{\Omega}_r \cap \bar{\Omega}_j$ is not an isolated vertex). This means that the domain $\tilde{\Xi}_{\mathrm{V},r}$ satisfying the conditions in **Definition 3.1** does not exist, so the vertex V does not satisfy the quasi-monotonicity assumption. Thus Γ_r does not contain an isolated vertex. In a similar way, we can explain that the set Γ_r does not contain an isolated edge. ♯

From now on, when we say two subdomains Ω_r and $\Omega_{r'}$ do not intersect if $\bar{\Omega}_r \cap \bar{\Omega}_{r'} = \emptyset$; otherwise we say the two subdomains intersect each other. For a polyhedron G in $\{\Omega_k\}$, we use Ξ_G to denote the union of all the polyhedra that belong to $\{\Omega_k\}$ and intersect with G. The following result will be used repeatedly.

Lemma 4.1 *Let G be a polyhedron in $\{\Omega_k\}$, and Γ be a union of some faces and edges of G. Then there exists an extension E_h mapping $Z_h(G)$ into $Z_h(\Omega)$ such that, for any function $\phi_h \in Z_h(G)$ vanishing on Γ, the function $E_h\phi_h$ satisfy the conditions: (1) $E_h\phi_h = \phi_h$ on G; (2) supp $E_h\phi_h \subset \Xi_G$; (3) when $G' \subset \Xi_G$ and $\bar{G}' \cap \bar{G} \subset \Gamma$, we have $E_h\phi_h = 0$ on G'; (4) the following stability estimates hold*

$$\|E_h\phi_h\|_{1,\Omega} \lesssim \log(1/h)\|\phi_h\|_{1,G} \quad \textit{and} \quad \|E_h\phi_h\|_{0,\Omega} \lesssim \|\phi_h\|_{0,G}. \tag{4.22}$$

Proof. We define the extension in the same manner as $\tilde{\mathbf{w}}_{h,1}$ in the proof of Theorem 3.1 in [18]. Assume that G has n_f faces, which are denoted by $\mathrm{F}_1, \cdots, \mathrm{F}_{n_f}$. Set $\mathrm{F}^\partial = \cup_{j=1}^{n_f} \partial\mathrm{F}_j$. For each F_j, let ϑ_{F_j} be the finite element function defined in [7] and [33]. This function satisfies $\vartheta_{\mathrm{F}_j}(\mathbf{x}) = 1$ for each node $\mathbf{x} \in \bar{\mathrm{F}}_j \backslash \partial\mathrm{F}_j$, $\vartheta_{\mathrm{F}_j}(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial G \backslash \mathrm{F}_j$ and $0 \leq \vartheta_{\mathrm{F}_j} \leq 1$ on G. Let π_h denote the standard interpolation operator into $Z_h(G)$, and define $\phi_h^{\mathrm{F}_j} = \pi_h(\vartheta_{\mathrm{F}_j}\phi_h)$ $(j = 1, \cdots, n_f)$.

When $\mathrm{F}_j \subset \Gamma$, we define the extension $\tilde{\phi}_h^{\mathrm{F}_j}$ of $\phi_h^{\mathrm{F}_j}$ as the natural zero extension since $\phi_h^{\mathrm{F}_j} = 0$ on ∂G. We need only to consider the faces $\mathrm{F}_j \not\subseteq \Gamma$. Let $G_j \subset \Xi_G$ be the polyhedron having the common face F_j with G. As in Lemma 4.5 of [22], we can show there exists an extension $\tilde{\phi}_h^{\mathrm{F}_j}$ of $\phi_h^{\mathrm{F}_j}$ such that $\tilde{\phi}_h^{\mathrm{F}_j} \in Z_h(\Omega)$; $\tilde{\phi}_h^{\mathrm{F}_j} = \phi_h^{\mathrm{F}_j}$ on $\bar{G}$; $\tilde{\phi}_h^{\mathrm{F}_j}$ vanishes on $\Omega \backslash (G \cup \mathrm{F}_j \cup G_j)$; $\tilde{\phi}_h^{\mathrm{F}_j}$ is stable with both H^1 norm and L^2 norm. Define

$$E_h\phi_h = \phi_h^\partial + \sum_{j=1}^{n_f} \tilde{\phi}_h^{\mathrm{F}_j},$$

where $\phi_h^\partial \in Z_h(\Omega)$ denotes the zero extension of the restriction of ϕ_h on F^∂. Then the extension $E_h\phi_h$ meet all the requirements in this lemma. $\sharp$

In the rest of this paper, for a nodal finite element function ϕ_h we always use $\tilde{\phi}_h$ to denote its extension defined by Lemma 4.1. For convenience, such an extension is simply called a *stable* extension.

4.1 A decomposition for edge element functions

In this subsection, we build a suitable Helmholtz decomposition for functions $\mathbf{v}_h \in V_h(\Omega)$. The basic ideas, which come from [19], can be described roughly as follows. We first divide all the subdomains $\{\Omega_r\}$ into groups according to the values of the coefficients $\{\alpha_r\}$, such that any two subdomains in each group do not intersect each other, and the subdomains in a former group correspond larger values of $\{\alpha_r\}$ than the subdomains in a later group. Then we in turn construct the desired decomposition from a former group to a later group.

As in [19], we decompose $\{\Omega_r\}_{r=1}^{N_0}$ into a union of non-empty subsets $\Sigma_1, \cdots, \Sigma_m$ satisfying the following conditions: (1) any two polyhedra in a same subset Σ_l do not intersect each other; (2) for any two polyhedra Ω_{r_l} and Ω_{r_j} belonging respectively to two different subsets Σ_l and Σ_j with $l < j$, we have $\alpha_{r_l} \geq \alpha_{r_j}$ if Ω_{r_l} and Ω_{r_j} intersect each other.

Without loss of generality, we assume that

$$\Sigma_l = \{\Omega_{n_{l-1}+1},\ \Omega_{n_{l-1}+2}, \cdots, \Omega_{n_l}\}$$

with $n_0 = 0$ and $n_l > n_{l-1}$ $(l = 1, \cdots, m)$. It is clear that Σ_l contains $(n_l - n_{l-1})$ polyhedra.

We are now ready to construct a desired decomposition for any $\mathbf{v}_h$ in $V_h(\Omega)$, and do so by three steps.

Step 1: Decompose $\mathbf{v}_h$ on all the polyhedra in Σ_1.

We shall write $\mathbf{v}_{h,r} = \mathbf{v}_h|_{\Omega_r}$. For $\Omega_r \in \Sigma_1$ (i.e., $1 \leq r \leq n_1$), by Theorem 3.1 of [18] we can decompose $\mathbf{v}_{h,r}$ as follows:

$$\mathbf{v}_{h,r} = \nabla p_{h,r} + \mathbf{r}_h \mathbf{w}_{h,r} + \mathbf{R}_{h,r}\,, \tag{4.1}$$

where $p_{h,r} \in Z_h(\Omega_r)$, $\mathbf{w}_{h,r} \in (Z_h(\Omega_r))^3$ and $\mathbf{R}_{h,r} \in V_h(\Omega_r)$, and they vanish on $\partial\Omega_r \cap \partial\Omega$. Moreover, we have

$$\|\mathbf{w}_{h,r}\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_{h,r}\|_{0,\Omega_r}, \quad \|\mathbf{w}_{h,r}\|_{0,\Omega_r} + \|p_{h,r}\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{v}_{h,r}\|_{0,\Omega_r} \tag{4.2}$$

and

$$h^{-1}\|\mathbf{R}_{h,r}\|_{0,\Omega_r} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_{h,r}\|_{0,\Omega_r}. \tag{4.3}$$

Let $\tilde{p}_{h,r} \in Z_h(\Omega)$ and $\tilde{\mathbf{w}}_{h,r} \in (Z_h(\Omega))^3$ be the *stable* extensions of $p_{h,r}$ and $\mathbf{w}_{h,r}$, respectively. Moreover, let $\tilde{\mathbf{R}}_{h,r} \in V_h(\Omega)$ denote the natural zero extensions of $\mathbf{R}_{h,r}$. Then we define

$$\tilde{\mathbf{v}}_{h,r} = \nabla\tilde{p}_{h,r} + \mathbf{r}_h\tilde{\mathbf{w}}_{h,r} + \tilde{\mathbf{R}}_{h,r} \ \text{ for all } r \text{ such that } \Omega_r \in \Sigma_1. \tag{4.4}$$

Step 2: Decompose $\mathbf{v}_h$ on all the polyhedra in Σ_2.

Consider a subdomain Ω_r from Σ_2. Without loss of generality, assume that there are only two polyhedra Ω_{r_1} and Ω_{r_2} in Σ_1 such that $\bar{\Omega}_{r_l} \cap \bar{\Omega}_r \neq \emptyset$. Set

$$\mathbf{v}^*_{h,r} = \mathbf{v}_{h,r} - (\tilde{\mathbf{v}}_{h,r_1} + \tilde{\mathbf{v}}_{h,r_2}) \ \text{ on } \Omega_r. \tag{4.5}$$

It is easy to see that $\lambda_e(\mathbf{v}^*_{h,r}) = 0$ for $e \subset \Gamma_r$. Then by **Proposition 4.1** and Theorem 3.1 in [18], there exist $p^*_{h,r} \in Z_h(\Omega_r)$, $\mathbf{w}^*_{h,r} \in (Z_h(\Omega_r))^3$ and $\mathbf{R}^*_{h,r} \in V_h(\Omega_r)$ having zero degrees of freedom on Γ_r such that

$$\mathbf{v}^*_{h,r} = \nabla p^*_{h,r} + \mathbf{r}_h \mathbf{w}^*_{h,r} + \mathbf{R}^*_{h,r} \quad \text{on} \quad \Omega_r\,. \tag{4.6}$$

Moreover, we have

$$\|\mathbf{w}^*_{h,r}\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}^*_{h,r}\|_{0,\Omega_r}, \quad \|\mathbf{w}^*_{h,r}\|_{0,\Omega_r} + \|p^*_{h,r}\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{v}^*_{h,r}\|_{0,\Omega_r} \tag{4.7}$$

and

$$h^{-1}\|\mathbf{R}^*_{h,r}\|_{0,\Omega_r} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}^*_{h,r}\|_{0,\Omega_r}. \tag{4.8}$$

Now we can define the decomposition of $\mathbf{v}_h$ on $\Omega_r \in \Sigma_2$ as

$$\mathbf{v}_{h,r} = \nabla(p^*_{h,r} + \sum_{l=1}^{2} \tilde{p}_{h,r_l}) + \mathbf{r}_h(\mathbf{w}^*_{h,r} + \sum_{l=1}^{2} \tilde{\mathbf{w}}_{h,r_l}) + \mathbf{R}^*_{h,r} + \sum_{l=1}^{2} \tilde{\mathbf{R}}_{h,r_l}\,. \tag{4.9}$$

Let $\tilde{p}^*_{h,r} \in Z_h(\Omega)$ and $\tilde{\mathbf{w}}^*_{h,r} \in (Z_h(\Omega))^3$ denote the stable extensions of $p^*_{h,r}$ and $\mathbf{w}^*_{h,r}$, respectively. Besides, let $\tilde{\mathbf{R}}^*_{h,r} \in V_h(\Omega)$ denote the standard extension of $\mathbf{R}^*_{h,r}$ by zero onto Ω. Since $p^*_{h,r}$, $\mathbf{w}^*_{h,r}$ and $\mathbf{R}^*_{h,r}$ have the zero degrees of freedom on Γ_r, by Lemma 5.1 the extensions $\tilde{p}^*_{h,r}$, $\tilde{\mathbf{w}}^*_{h,r}$ and $\tilde{\mathbf{R}}^*_{h,r}$ vanish on every $\Omega_l \in \Sigma_1$. Then we set

$$\tilde{\mathbf{v}}^*_{h,r} = \nabla \tilde{p}^*_{h,r} + \mathbf{r}_h \tilde{\mathbf{w}}^*_{h,r} + \tilde{\mathbf{R}}^*_{h,r} \quad \text{for all } r \text{ such that} \quad \Omega_r \in \Sigma_2. \tag{4.10}$$

Step 3: Obtain the final desired decomposition of $\mathbf{v}_h$.

We now consider the index $l \geq 3$, and assume that the decompositions of $\mathbf{v}_h$ on all polyhedra belonging to Σ_1 $\Sigma_2, \cdots, \Sigma_{l-1}$ are done as in Steps 1 and 2. Next, we will build up a decomposition of $\mathbf{v}_h$ in all subdomains $\Omega_r \in \Sigma_l$.

For the ease of notation, we introduce two index sets:

$$\Lambda^1_r = \{\ i\ ;\ 1 \leq i \leq n_1 \text{ such that} \ \ \partial\Omega_i \cap \partial\Omega_r \neq \emptyset\}\,,$$

$$\Lambda^{l-1}_r = \{\ i\ ;\ n_1 + 1 \leq i \leq n_{l-1} \text{ such that} \ \ \partial\Omega_i \cap \partial\Omega_r \neq \emptyset\}.$$

Define

$$\mathbf{v}^*_{h,r} = \mathbf{v}_{h,r} - \sum_{i \in \Lambda^1_r} \tilde{\mathbf{v}}_{h,i} - \sum_{i \in \Lambda^{l-1}_r} \tilde{\mathbf{v}}^*_{h,i} \quad \text{on} \quad \Omega_r. \tag{4.11}$$

By the definitions of $\tilde{\mathbf{v}}_{h,i}$ and $\tilde{\mathbf{v}}^*_{h,i}$, we know $\lambda_e(\mathbf{v}^*_{h,r}) = 0$ for all $e \subset \Gamma_r$. So by **Proposition 4.1** and Theorem 3.1 in [18], one can find $p^*_{h,r} \in Z_h(\Omega_r)$, and $\mathbf{w}^*_{h,r} \in (Z_h(\Omega_r))^3$ and $\mathbf{R}^*_{h,r} \in V_h(\Omega_r)$ such that

$$\mathbf{v}^*_{h,r} = \nabla p^*_{h,r} + \mathbf{r}_h \mathbf{w}^*_{h,r} + \mathbf{R}^*_{h,r} \quad \text{on} \quad \Omega_r\,, \tag{4.12}$$

and they have the zero degrees of freedom on Γ_r. Moreover, we have

$$\|\mathbf{w}^*_{h,r}\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}^*_{h,r}\|_{0,\Omega_r}, \quad \|\mathbf{w}^*_{h,r}\|_{0,\Omega_r} + \|p^*_{h,r}\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{v}^*_{h,r}\|_{0,\Omega_r} \tag{4.13}$$

and

$$h^{-1}\|\mathbf{R}^*_{h,r}\|_{0,\Omega_r} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}^*_{h,r}\|_{0,\Omega_r}. \tag{4.14}$$

Using (4.11) and (4.12), we have the following decomposition for $\mathbf{v}_h$ on each $\Omega_r \in \Sigma_l$:

$$\mathbf{v}_{h,r} \;\; = \;\; \nabla(p^*_{h,r} + \sum_{i \in \Lambda^1_r} \tilde{p}_{h,i} + \sum_{i \in \Lambda^{l-1}_r} \tilde{p}^*_{h,i}) + \;\mathbf{w}^*_{h,r} + \sum_{i \in \Lambda^1_r} \tilde{\mathbf{w}}_{h,i} + \sum_{i \in \Lambda^{l-1}_r} \tilde{\mathbf{w}}^*_{h,i}$$

$$+ \quad \mathbf{R}^*_{h,r} + \sum_{i\in\Lambda^1_r} \tilde{\mathbf{R}}_{h,i} + \sum_{i\in\Lambda^{l-1}_r} \tilde{\mathbf{R}}^*_{h,i} \quad \text{on} \quad \Omega_r. \tag{4.15}$$

As it was done in Steps 1 and 2, we can extend $p^*_{h,r}$, $\mathbf{w}^*_{h,r}$ and $\mathbf{R}^*_{h,r}$ onto the entire domain Ω to get $\tilde{p}^*_{h,r}$, $\tilde{\mathbf{w}}^*_{h,r}$ and $\tilde{\mathbf{R}}^*_{h,r}$. Then, by Lemma 5.1, the extensions $\tilde{p}^*_{h,r}$, $\tilde{\mathbf{w}}^*_{h,r}$ and $\tilde{\mathbf{R}}^*_{h,r}$ vanish on every $\Omega_i \in \Sigma_j$ for $1 \le j \le l-1$. Then we define

$$\tilde{\mathbf{v}}^*_{h,r} = \nabla \tilde{p}^*_{h,r} + \mathbf{r}_h \tilde{\mathbf{w}}^*_{h,r} + \tilde{\mathbf{R}}^*_{h,r} \quad \text{for all } r \text{ such that } \Omega_r \in \Sigma_l. \tag{4.16}$$

It is clear that $\lambda_e(\tilde{\mathbf{v}}^*_{h,r}) = 0$ for all $e \in \Gamma_r$.

Continuing with the above procedure for all l's till $l = m$, we will have built up the decomposition of $\mathbf{v}_h$ over all the subdomains $\Omega_1, \Omega_2, \ldots, \Omega_{N_0}$ such that

$$\mathbf{v}_h = \sum_{r=1}^{n_1} \tilde{\mathbf{v}}_{h,r} + \sum_{r=n_1+1}^{n_m} \tilde{\mathbf{v}}^*_{h,r} = \nabla p_h + \mathbf{r}_h \mathbf{w}_h + \mathbf{R}_h \tag{4.17}$$

where $p_h \in Z_h(\Omega)$, $\mathbf{w}_h \in (Z_h(\Omega))^3$ and $\mathbf{R}_h \in V_h(\Omega)$ are given by

$$p_h = \sum_{r=1}^{n_1} \tilde{p}_{h,r} + \sum_{r=n_1+1}^{n_m} \tilde{p}^*_{h,r}, \quad \mathbf{w}_h = \sum_{r=1}^{n_1} \tilde{\mathbf{w}}_{h,r} + \sum_{r=n_1+1}^{n_m} \tilde{\mathbf{w}}^*_{h,r} \tag{4.18}$$

and

$$\mathbf{R}_h = \sum_{r=1}^{n_1} \tilde{\mathbf{R}}_{h,r} + \sum_{r=n_1+1}^{n_m} \tilde{\mathbf{R}}^*_{h,r}. \tag{4.19}$$

Remark 4.1 *We would like to emphasize that, for each $\Omega_r \in \Sigma_l$ ($2 \le l \le m$), the extensions $\tilde{p}^*_{h,r}$, $\tilde{\mathbf{w}}^*_{h,r}$ and $\tilde{\mathbf{R}}^*_{h,r}$ vanish on every $\Omega_i \in \Sigma_j$ for $1 \le j \le l-1$. Otherwise, the decomposition (4.17) do not valid yet. This is why we have to, in Theorem 3.1 (and Theorem 4.1) in [18], require that $\mathbf{w}_h$, p_h (and $\mathbf{R}_h$) vanish on Γ. This is also the reason that we have to build various "complex" decompositions and have not simply used the standard regular Helmholtz decomposition in each subdomain Ω_r. For example, we can not require the function Φ defined in the standard regular Helmholtz decomposition vanishes on an edge* E *(comparing Lemma 3.1 in [18]).*

4.2 Stability of the decomposition

In this subsection, we are devoted to the proof of the stability estimates in Theorem 3.1 based on the Helmholtz decomposition defined in the previous subsection.

For a polyhedron $\Omega_r \in \Sigma_l$ ($l \ge 2$), define

$$\Lambda^{(j)}_r(a) = \{i;\ \Omega_i \in \Sigma_j \ \text{ and } \ \bar{\Omega}_i \cap \bar{\Omega}_r \ne \emptyset\} \quad (1 \le j \le l-1).$$

Notice that $\Lambda^{(j)}_r(a)$ may be an empty set for some j. For such Ω_r, we use $L_r(a)$ to denote the number of all the non-empty sets $\Lambda^{(j)}_r(a)$. Without loss of generality, we assume that the sets $\Lambda^{(j)}_r(a)$ are non-empty for $j = 1, \cdots, L_r(a)$.

As in [19], we can prove the following auxiliary result by induction, together with (4.2)-(4.3), (4.7)-(4.8) and (4.13)-(4.14).

Lemma 4.2 *For any subdomain Ω_r from Σ_l ($l \geq 2$), let $\mathbf{v}^*_{h,r}$ be defined as in Steps 2 and 3 for the construction of the decomposition of any $\mathbf{v}_h \in V_h(\Omega)$ in Subsection 4.1. Then $\mathbf{v}^*_{h,r}$ admits the following estimates*

$$\|\mathbf{curl}\,\mathbf{v}^*_{h,r}\|_{0,\Omega_r} \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_r} + \sum_{j=1}^{L_r(a)} \log^{2j}(1/h) \sum_{i\in\Lambda_r^{(j)}(a)} \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i} \tag{4.20}$$

and

$$\|\mathbf{v}^*_{h,r}\|_{0,\Omega_r} \lesssim \|\mathbf{v}_h\|_{0,\Omega_r} + \sum_{j=1}^{L_r(a)} \log^{2j}(1/h) \sum_{i\in\Lambda_r^{(j)}(a)} \|\mathbf{v}_h\|_{\mathbf{curl},\Omega_i}. \tag{4.21}$$

$\sharp$

Proof of Theorem 3.1. We are now ready to show Theorem 3.1. We start with the estimate of $\|\mathbf{w}_h\|^2_{H^1_\alpha(\Omega_r)}$ for each subdomain Ω_r in Σ_1, i.e., $1 \leq r \leq n_1$. For such case, we have $\mathbf{w}_h|_{\Omega_r} = \mathbf{w}_{h,r}$ with $\mathbf{w}_{h,r}$ being defined by (4.1) (note that any two of the subdomains $\Omega_1, \cdots, \Omega_{n_1}$ do not intersect). Then, it follows by (4.2) that

$$\|\mathbf{w}_h\|^2_{H^1_\alpha(\Omega_r)} \lesssim \log^2(1/h)\|\alpha^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_{h,r}\|^2_{0,\Omega_r} = \log^2(1/h)\|\alpha^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|^2_{0,\Omega_r}. \tag{4.22}$$

Next, we consider all the subdomains Ω_r in Σ_2. As in Step 2 of the construction of the stable decomposition for $\mathbf{v}_h$, we assume that Ω_r intersects only two subdomains Ω_{r_1} and Ω_{r_2} in Σ_1. Then we have

$$\mathbf{w}_h|_{\Omega_r} = \tilde{\mathbf{w}}^*_{h,r} + (\tilde{\mathbf{w}}_{h,r_1} + \tilde{\mathbf{w}}_{h,r_2})|_{\Omega_r}.$$

By the triangle inequality,

$$\|\mathbf{w}_h\|_{1,\Omega_r} \lesssim \|\tilde{\mathbf{w}}^*_{h,r}\|_{1,\Omega_r} + \|\tilde{\mathbf{w}}_{h,r_1}\|_{1,\Omega_r} + \|\tilde{\mathbf{w}}_{h,r_2}\|_{1,\Omega_r}. \tag{4.23}$$

Since $\tilde{\mathbf{w}}^*_{h,r} = \mathbf{w}^*_{h,r}$ on Ω_r, we get by (4.7)

$$\|\tilde{\mathbf{w}}^*_{h,r}\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}^*_{h,r}\|_{0,\Omega_r}.$$

Plugging this inequality and (4.2) (with $r = r_1, r_2$) in (4.23), and using (4.20) for $l = 2$, leads to

$$\|\mathbf{w}_h\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_r} + \log^3(1/h)\sum_{j=1}^{2}\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_{r_j}}. \tag{4.24}$$

Then by inserting the coefficient α, we readily have for all subdomains $\Omega_r \in \Sigma_2$ that

$$\begin{aligned}\alpha_r^{\frac{1}{2}}\|\mathbf{w}_h\|_{1,\Omega_r} &\lesssim \log(1/h)\|\alpha_r^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_r} + \log^3(1/h)\sum_{j=1}^{2}\|\alpha_r^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_{r_j}} \\ &\lesssim \log(1/h)\|\alpha_r^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_r} + \log^3(1/h)\sum_{j=1}^{2}\|\alpha_{r_j}^{\frac{1}{2}}\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_{r_j}}. \end{aligned} \tag{4.25}$$

Here we have used the fact that $\alpha_r \leq \alpha_{r_j}$, which comes from the definitions of Σ_1 and Σ_2.

Finally we consider all the subdomains Ω_r from the general class Σ_l with $l \geq 3$. By the definition of $\mathbf{w}_h$ (see (4.18)), we have for $\mathbf{w}_h$ in Ω_r

$$\mathbf{w}_h = \mathbf{w}^*_{h,r} + \sum_{i\in\Lambda_r^1}\tilde{\mathbf{w}}_{h,i} + \sum_{i\in\Lambda_r^{l-1}}\tilde{\mathbf{w}}^*_{h,i}. \tag{4.26}$$

Then

$$\|\mathbf{w}_h\|_{1,\Omega_r} \lesssim \|\mathbf{w}^*_{h,r}\|_{1,\Omega_r} + \sum_{i\in\Lambda^1_r} \|\tilde{\mathbf{w}}_{h,i}\|_{1,\Omega_r} + \sum_{i\in\Lambda^{l-1}_r} \|\tilde{\mathbf{w}}^*_{h,i}\|_{1,\Omega_r}. \quad (4.27)$$

By Lemma 4.1, we have

$$\|\tilde{\mathbf{w}}_{h,i}\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{w}_{h,i}\|_{1,\Omega_i} \quad (i \in \Lambda^1_r)$$

and

$$\|\tilde{\mathbf{w}}^*_{h,i}\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{w}^*_{h,i}\|_{1,\Omega_i} \quad (i \in \Lambda^{l-1}_r).$$

This, together with (4.2) and (4.7), leads to

$$\|\tilde{\mathbf{w}}_{h,i}\|_{1,\Omega_r} \lesssim \log^2(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{1,\Omega_i} \quad (i \in \Lambda^1_r)$$

and

$$\|\tilde{\mathbf{w}}^*_{h,i}\|_{1,\Omega_r} \lesssim \log^2(1/h)\|\mathbf{curl}\,\mathbf{v}^*_{h,i}\|_{1,\Omega_i} \quad (i \in \Lambda^{l-1}_r).$$

Substituting (4.13) and the above two inequalities into (4.27), yields

$$\begin{aligned}\|\mathbf{w}_h\|_{1,\Omega_r} &\lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}^*_{h,r}\|_{1,\Omega_r} \\ &+ \log^2(1/h)\sum_{i\in\Lambda^1_r}\|\mathbf{curl}\,\mathbf{v}_h\|_{1,\Omega_i} + \log^2(1/h)\sum_{i\in\Lambda^{l-1}_r}\|\mathbf{curl}\,\mathbf{v}^*_{h,i}\|_{1,\Omega_i}. \quad (4.28)\end{aligned}$$

But, from (4.20) we have

$$\|\mathbf{curl}\,\mathbf{v}^*_{h,i}\|_{0,\Omega_i} \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i} + \sum_{j=1}^{L_i(a)} \log^{2j}(1/h) \sum_{k\in\Lambda^{(j)}_r(a)} \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_k} \ \ (i = r \ \text{ or } \ i \in \Lambda^{l-1}_r).$$

Then we further deduce from (4.28) that

$$\|\mathbf{w}_h\|_{1,\Omega_r} \lesssim \log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_r} + \sum_{j=1}^{L_r(a)} \log^{2j+1}(1/h) \sum_{i\in\Lambda^{(j)}_r(a)} \|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i}.$$

Inserting the coefficient α_r in the above inequality and using the relation $\alpha_r \leq \alpha_i$ (for $i \in \Lambda^{(j)}_r(a)$) gives

$$\begin{aligned}\alpha_r^{\frac{1}{2}}\|\mathbf{w}_h\|_{1,\Omega_r} &\lesssim \log(1/h)\|\alpha_r^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_r} \\ &\quad + \sum_{j=1}^{L_r(a)} \log^{2j+1}(1/h) \sum_{i\in\Lambda^{(j)}_r(a)} \|\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i}.\end{aligned}$$

It is clear that the set $\Lambda^{(j)}_r(a)$ contains only a few indices i and $L_r(a)$ is a finite number. Summing up the above estimate with the ones in (4.22) and (4.25), we obtain

$$\begin{aligned}\sum_{r=1}^{N_0}\alpha_r\|\mathbf{w}_h\|^2_{1,\Omega_r} &\lesssim \log^2(1/h)\|\alpha_r^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|^2_{0,\Omega} \\ &\quad + \sum_{r=n_1+1}^{N_0}\sum_{j=1}^{L_r(a)} \log^{2(2j+1)}(1/h) \sum_{i\in\Lambda^{(j)}_r(a)} \|\alpha_i^{\frac{1}{2}}\mathbf{curl}\ \mathbf{v}_h\|^2_{0,\Omega_i}\end{aligned}$$

$$\lesssim \log^{2m}(1/h) \sum_{r=1}^{N_0} \|\alpha_r^{\frac{1}{2}} \mathbf{curl}\ \mathbf{v}_h\|^2_{0,\Omega_r}, \tag{4.29}$$

where $m = \max_{1\leq r\leq N_0}(2L_r(a)+1)$. It follows by (4.29) that

$$(\sum_{r=1}^{N_0} \alpha_r \|\mathbf{w}_h\|^2_{1,\Omega_r})^{\frac{1}{2}} \lesssim \log^m(1/h)\|\alpha^{\frac{1}{2}} \mathbf{curl}\, \mathbf{v}_h\|_{0,\Omega}.$$

In an analogous way, but using (4.21) and **Assumption 3.1**, we can verify

$$\|\beta^{\frac{1}{2}}\mathbf{w}_h\|_{0,\Omega}, \quad \|p_h\|_{H^1_\beta(\Omega)} \lesssim \|\beta^{\frac{1}{2}}\mathbf{v}_h\|_{0,\Omega}.$$

The term $\|\mathbf{R}_h\|^2_{L^2_\alpha(\Omega_r)}$ can be estimated more easily. This completes the proof of Theorem 3.1. $\sharp$

Remark 4.2 *In the proof of Theorem 3.1, we need not to use* **Assumption 3.2**. *This thanks the L^2 stability estimates in (4.2), (4.7) and (4.13), which come from the second estimate in Theorem 3.1 of [18].*

5 Analysis for the general case

In this section, we give an exact definition of the space $V_h^*(\Omega)$ in Subsubsection 3.3.2 and present a proof of Theorem 3.2.

Let $\Im^*_{\rm V}$ and $\Im^c_{\rm V}$ be the two sets defined in **Remark 3.1** for a *weird vertex* V. Define $\Im^*_s = \cup_{{\rm V}\in\mathcal{V}_s}\Im^*_{\rm V}$ and $\Im^c_s = \cup_{{\rm V}\in\mathcal{V}_s}\Im^c_{\rm V}$. When there is no weird vertex, we have $\Im^c = \emptyset$. In general there are some non-weird vertices in $\mathcal{N}(\Omega)\cup\mathcal{N}(\partial\Omega)$, i.e., $\tilde{\Im} = \{\Omega_k\}_{k=1}^{N_0}\backslash(\Im^*_s\cup\Im^c_s) \neq \emptyset$.

Let $\Gamma_r \subset \partial\Omega_r$ be the set defined at the beginning of Section 4. This set has the following property

Proposition 5.1. For each $\Omega_r \in \Im^*_s \cup \tilde{\Im}$, the set Γ_r does not contain any isolated vertex, i.e., Γ_r is a union of faces and edges only.

Proof. Let $\Omega_r \in \Im^*_s$ and V be an isolated vertex in Γ_r. By the definition of Γ_r, there is a polyhedron Ω_l satisfying $\alpha_l \geq \alpha_r$ such that $\bar{\Omega}_r \cap \bar{\Omega}_l = {\rm V}$. On the other hand, since $\Omega_r \in \Im^*_{\rm V}$, by the definition of $\Im^*_{\rm V}$ there exists another polyhedron $\Omega_{l'}$ satisfying $\alpha_{l'} \geq \alpha_r$ such that the intersection $\bar{\Omega}_r \cap \bar{\Omega}_{l'}$ contains the vertex V, but it is a face or an edge of Ω_r. This means that V is not an isolated vertex, which belongs to the common face or common edge of Ω_r and $\Omega_{l'}$. When $\Omega_r \in \tilde{\Im}$, the conclusion can be proved in a similar way since each vertex of Ω_r is not a weird vertex. $\sharp$

If there is no weird vertex, then we have $\Im^c_{\rm V} = \emptyset$ and so $\Im^*_s \cup \tilde{\Im} = \{\Omega_k\}_{k=1}^{N_0}$, which means that **Proposition 5.1** holds for all polyhedra Ω_r. Then, by **Proposition 5.1** and Theorem 4.1 in [18], we can establish the desired Helmholtz decomposition as in Section 4. This inspires us to construct discrete Helmholtz decompositions on the polyhedra belonging to $\Im^c_s$ and $\Im^*_s \cup \tilde{\Im}$ separately. For simplicity of exposition, we assume that each polyhedron $\Omega_r \in \tilde{\Im}$ satisfies the condition: when $\bar{\Omega}_r \cap \bar{\Omega}_{r'} \neq \emptyset$ for some $\Omega_{r'} \in \Im^c_s$, we have $\alpha_r \leq \alpha_{r'}$. Under the assumption, we can first construct a desired Helmholtz decomposition on all the polyhedra in $\Im^c_s$ and then do this on the polyhedra in $\Im^*_s\cup\tilde{\Im}$. Otherwise, we need to consider the polyhedra that do not satisfy this condition before building the decomposition on the polyhedra in $\Im^c_s$ but will not change the main ideas in the analysis.

We use the same notations as in Subsubsection 3.3.2. Without loss of generality, we assume that $n_{\rm V} = 2$ for any ${\rm V} \in \mathcal{V}_s^{in}$ and $n_{\rm V} = 1$ for any ${\rm V} \in \mathcal{V}_s^b$. Then n_s just equals the number of all the weird vertices ($1 \le n_s \le N_0-1$). For more general case, we need to slightly change definitions of the functionals $\{\mathcal{F}_l\}$ (refer to Theorem 5.3 in [18]). Furthermore, we assume that

$$\mathcal{V}_s^{in} = \{\mathrm{V}_1, \cdots, \mathrm{V}_{n_s-1}\} \quad \text{and} \quad \mathcal{V}_s^{b} = \{\mathrm{V}_{n_s}\}.$$

For convenience, we write all the polyhedrons in $\Im_s^c$ as $\Omega_1, \cdots, \Omega_{n_s}$ and assume that Ω_i and Ω_{i+1} are two neighboring polyhedra with $\bar{\Omega}_i \cap \bar{\Omega}_{i+1} = \mathrm{V}_i$ ($i = 1, \cdots, n_s - 1$) and $\bar{\Omega}_{n_s} \cap \partial\Omega = \mathrm{V}_{n_s}$, but $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$ if $|j - i| \ge 2$. Let D_s be the union of all the polyhedra in $\Im_s^c$, i.e., $D_s = \cup_{i=1}^{n_s}\bar{\Omega}_i$, then $D_s \subset \Omega$ is a non-Lipchitz domain.

For Ω_1, we choose a face F_1 satisfying $\mathrm{V}_1 \in \mathrm{F}_1 \subset \partial\Omega_1$ and define functions $C_{\partial\mathrm{F}_1}$ and $\phi_{\partial\mathrm{F}_1}$ as

$$C_{\partial\mathrm{F}_1} = \frac{1}{l_1}\int_0^{l_i} (\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}_1})(s)\, ds, \quad \phi_{\partial\mathrm{F}_1}(t) = \int_0^{t} (\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}_1} - C_{\partial\mathrm{F}_1})(s) ds + c_1\,, \quad \forall\, t \in [0, l_1]\,,$$

where l_1 is the length of $\partial\mathrm{F}_1$ and $t = 0$ (and $t = l_1$) corresponds the vertex V_1. The constant c_1 is chosen such that $\gamma_{\mathrm{E}}(\phi_{\partial\mathrm{F}_1}) = 0$, where $\mathrm{E} \subset \partial\mathrm{F}_1$ is an edge or a union of several edges.

But, for each Ω_i with $i = 2, \cdots, n_s$, in general we need to choose two faces $\mathrm{F}_i^{(i-1)}$ and $\mathrm{F}_i^{(i)}$ satisfying $\mathrm{V}_l \in \mathrm{F}_i^{(l)} \subset \partial\Omega_i$ ($l = i-1, i$) except that V_{i-1} and V_i is just two vertices of a same face of Ω_i, where we need to replace $\mathrm{F}_i^{(i-1)}$ and $\mathrm{F}_i^{(i)}$ by this face. Similarly, we define functions $C_{\partial\mathrm{F}_i^{(l)}}$ and $\phi_{\partial\mathrm{F}_i^{(l)}}$ ($l = i-1, i$) by

$$C_{\partial\mathrm{F}_i^{(l)}} = \frac{1}{l_1}\int_0^{l_i^{(l)}} (\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}_i^{(l)}})(s)\, ds, \quad \phi_{\partial\mathrm{F}_i^{(l)}}(t) = \int_0^{t} (\mathbf{v}_h \cdot \mathbf{t}_{\partial\mathrm{F}_i^{(l)}} - C_{\partial\mathrm{F}_i^{(l)}})(s) ds + c_i^{(l)}\,, \ \forall\, t \in [0, l_i^{(l)}]\,,$$

where $l_i^{(l)}$ is the length of $\partial\mathrm{F}_i^{(l)}$ and $t = 0$ (and $t = l_i^{(l)}$) corresponds the vertex V_l ($l = i-1, i$). The constant $c_i^{(l)}$ is chosen such that $\gamma_{\mathrm{E}}(\phi_{\partial\mathrm{F}_i^{(l)}}) = 0$ with $\mathrm{E} \subset \partial\mathrm{F}_i^{(l)}$ being an edge or a union of some edges.

Now we define the functional as follows

$$\mathcal{F}_1\mathbf{v}_h = \phi_{\partial\mathrm{F}_2^{(1)}}(\mathrm{V}_1) - \phi_{\partial\mathrm{F}_1}(\mathrm{V}_1), \quad \mathcal{F}_i\mathbf{v}_h = \phi_{\partial\mathrm{F}_{i+1}^{(i)}}(\mathrm{V}_i) - \phi_{\partial\mathrm{F}_i^{(i)}}(\mathrm{V}_i) \ \ (i = 2, \cdots, n_s - 1)$$

and $\mathcal{F}_{n_s}\mathbf{v}_h = \phi_{\partial\mathrm{F}_{n_s}^{(n_s)}}(\mathrm{V}_{n_s})$. With these functionals, we define the space

$$V_h^*(\Omega) = \{\mathbf{v}_h \in V_h(\Omega) : \ \mathcal{F}_l\mathbf{v}_h = 0 \ \text{ for } \ l = 1, \cdots, n_s\}.$$

As pointed out in Subsubsection 3.3.2, the key ingredient is the codimension n_s of the space $V_h^*(\Omega)$ instead of the space itself. In fact, there may be different choices of the functionals $\{\mathcal{F}_i\}$ in the definition of this space.

Proof of Theorem 3.2. The proof is divided into three steps.
Step 1. Define a decomposition of $\mathbf{v}_h \in V_h^*(\Omega)$ on the non-Lipchitz domain D_s.

As in the proof of Lemma 4.3 in [18], we define extension functions $\tilde{C}_{\mathrm{V}_1}$ and $\tilde{\phi}_{\mathrm{V}_1}$ of $C_{\partial\mathrm{F}_1}$ and $\phi_{\partial\mathrm{F}_1}$ (regarding Ω_1 and V_1 as G and V in that lemma). Similarly, we define the extensions $\tilde{\phi}_{\mathrm{V}_l}^{(i)}$ and $\tilde{C}_{\mathrm{V}_l}^{(i)}$ of $\phi_{\partial\mathrm{F}_i^{(l)}}$ and $C_{\partial\mathrm{F}_i^{(l)}}$ ($l = i-1, i$), respectively. Here, the values of these extensions vanish at all the nodes in Ω_i except on $\partial\mathrm{F}_i^{(l)}$ and we have $\tilde{C}_{\mathrm{V}_l}^{(i)} = \mathbf{0}$ at V_l ($l = i-1, i$). Define

$$\hat{\mathbf{v}}_h^{(1)} = \mathbf{v}_h|_{\Omega_1} - (\nabla\tilde{\phi}_{\mathrm{V}_1} + \mathbf{r}_h\tilde{C}_{\mathrm{V}_1}) \quad \text{on} \ \ \Omega_1 \tag{5.1}$$

and

$$\hat{\mathbf{v}}_h^{(i)} = \mathbf{v}_h|_{\Omega_i} - (\nabla\tilde{\phi}^{(i)}_{\mathrm{V}_{i-1}} + \mathbf{r}_h\tilde{C}^{(i)}_{\mathrm{V}_{i-1}}) - (\nabla\tilde{\phi}^{(i)}_{\mathrm{V}_i} + \mathbf{r}_h\tilde{C}^{(i)}_{\mathrm{V}_i}) \quad \text{on} \quad \Omega_i \quad (i = 2, \cdots, n_s). \tag{5.2}$$

Then we have $\lambda_e(\hat{\mathbf{v}}_h^{(1)}) = 0$ for any $e \subset \partial\mathrm{F}_1$ by the definitions of $\tilde{\phi}_{\mathrm{V}_1}$ and $\tilde{C}_{\mathrm{V}_1}$. Similarly, we have $\lambda_e(\hat{\mathbf{v}}_h^{(i)}) = 0$ for any $e \subset \partial\mathrm{F}_i^{(l)}$ $(i = 2, \cdots, n_s; l = i-1, i)$. Thus we can use **Corollary 4.1** in [18] for $\hat{\mathbf{v}}_h^{(i)}$ to build a decomposition of $\mathbf{v}_h|_{\Omega_i}$ as

$$\hat{\mathbf{v}}_h^{(i)} = \nabla p_h^{(i)} + \mathbf{r}_h\mathbf{w}_h^{(i)} + \mathbf{R}_h^{(i)} \quad \text{on} \quad \Omega_i \quad (i = 1, 2, \cdots, n_s), \tag{5.3}$$

where the finite element functions $p_h^{(i)}$, $\mathbf{w}_h^{(i)}$ and $\mathbf{R}_h^{(i)}$ have the zero degrees of freedom on $\partial\mathrm{F}_1$ (for $i = 1$) or on $\partial\mathrm{F}_i^{(l)}$ $(i = 2, \cdots, n_s; l = i-1, i)$. Define a function p_h^s as follows

$$p_h^s = p_h^{(1)} + \tilde{\phi}_{\mathrm{V}_1} \quad \text{on} \quad \Omega_1; \qquad p_h^s = p_h^{(i)} + \tilde{\phi}^{(i)}_{\mathrm{V}_{i-1}} + \tilde{\phi}^{(i)}_{\mathrm{V}_i} \quad \text{on} \quad \Omega_i \quad (i = 2, \cdots, n_s).$$

Similarly, we can define two functions $\mathbf{w}_h^s$ and $\mathbf{R}_h^s$. Then we have $p_h^s|_{\Omega_i} \in Z_h(\Omega_i)$, $\mathbf{w}_h^s|_{\Omega_i} \in (Z_h(\Omega_i))^3$ and $\mathbf{R}_h^s|_{\Omega_i} \in V_h(\Omega_i)$ for $i = 1, 2, \cdots, n_s$. Moreover, the function $\mathbf{w}_h^s$ vanishes at all the vertices $\{\mathrm{V}_i\}$ by its definition, and the function p_h^s vanishes at V_{n_s} and is continuous at the vertices $\mathrm{V}_1, \cdots, \mathrm{V}_{n_s-1}$ by the assumption $\mathbf{v}_h \in V_h^*(\Omega)$. Combining (5.1)-(5.3), we get a decomposition

$$\mathbf{v}_h = \nabla p_h^s + \mathbf{r}_h\mathbf{w}_h^s + \mathbf{R}_h^s \quad \text{on} \quad D_s. \tag{5.4}$$

Moreover, the following estimates hold by the proof of Lemma 4.3 of [18]

$$\|p_h^s\|_{1,\Omega_i} \leq C\log(1/h)\|\mathbf{v}_h\|_{\mathbf{curl},\Omega_i}, \tag{5.5}$$

$$\|\mathbf{w}_h^s\|_{1,\Omega_i} \leq C\log(1/h)\|\mathbf{curl}\ \mathbf{v}_h\|_{0,\Omega_i} \tag{5.6}$$

and

$$h^{-1}\|\mathbf{R}_h^s\|_{0,\Omega_i} \leq C\log(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{0,\Omega_i} \tag{5.7}$$

for $i = 1, \cdots, n_s$. Here, the complete norm $\|p_h^s\|_{1,\Omega_i}$ is bounded with a logarithmic factor thanks the assumption $\mathbf{v}_h \in V_h^*(\Omega)$.

Step 2. Build a decomposition of $\mathbf{v}_h \in V_h^*(\Omega)$ on the global domain Ω.

For a polyhedron $G \in \Im_s^* \cup \tilde{\Im}$, define an extension $\tilde{\mathbf{w}}_h^s|_G$ as follows: $\tilde{\mathbf{w}}_h^s = \mathbf{w}_h^s$ on $\cup_{i=1}^{n_s}\bar{\Omega}_i$; $\tilde{\mathbf{w}}_h^s$ vanishes at all the nodes on $\partial G \backslash \cup_{i=1}^{n_s}\bar{\Omega}_i$; $\tilde{\mathbf{w}}_h^s$ is discrete harmonic in G. It is clear that, when $G \cap (\cup_{i=1}^{n_s}\bar{\Omega}_i) = \emptyset$, we have $\tilde{\mathbf{w}}_h^s = \mathbf{0}$ on G. Similarly, we can define an extension $\tilde{p}_h^s$ of p_h^s. Let $\tilde{\mathbf{R}}_h^s$ be the natural zero extension of $\mathbf{R}_h^s$. Then $\tilde{p}_h^s \in Z_h(\Omega)$, $\tilde{\mathbf{w}}_h^s \in (Z_h(\Omega))^3$ and $\tilde{\mathbf{R}}_h^s \in V_h(\Omega)$. Define

$$\mathbf{v}_h^* = \mathbf{v}_h - (\nabla\tilde{p}_h^s + \mathbf{r}_h\tilde{\mathbf{w}}_h^s + \tilde{\mathbf{R}}_h^s) \quad \text{on} \quad \Omega. \tag{5.8}$$

Then $\mathbf{v}_h^*$ vanishes on D_s. As in Section 4, we can build a decomposition of $\mathbf{v}_h^*$ on $\Omega\backslash D_s$ by **Proposition 5.1** and Theorem 4.1 of [18] (but here we cannot use Theorem 3.1 in [18] since edges may be contained in Γ_r)

$$\mathbf{v}_h^* = \nabla p_h^* + \mathbf{r}_h\mathbf{w}_h^* + \mathbf{R}_h^* \quad \text{on} \quad \Omega\backslash D_s, \tag{5.9}$$

where p_h^*, $\mathbf{w}_h^*$ and $\mathbf{R}_h^*$ have the zero degrees of freedom on $\bar{G} \cap D_s$ for each $G \in \Im_s^* \cup \tilde{\Im}$. Then we can naturally define their zero extension functions $\tilde{p}_h^*$, $\tilde{\mathbf{w}}_h^*$ and $\tilde{\mathbf{R}}_h^*$ and have the decomposition by (5.9)

$$\mathbf{v}_h^* = \nabla\tilde{p}_h^* + \mathbf{r}_h\tilde{\mathbf{w}}_h^* + \tilde{\mathbf{R}}_h^* \quad \text{on} \quad \Omega. \tag{5.10}$$

Moreover, the following estimates hold

$$\|\tilde{p}_h^*\|_{H^1_\beta(\Omega)} \leq C\log^{m'}(1/h)\|\mathbf{v}_h^*\|_{H^*(\mathbf{curl},\Omega)}, \tag{5.11}$$

$$\|\tilde{\mathbf{w}}_h^*\|_{H^1_*(\Omega)} \leq C\log^{m'}(1/h)\|\mathbf{curl}\,\mathbf{v}_h^*\|_{L^2_\alpha(\Omega)} \tag{5.12}$$

and

$$h^{-1}\|\tilde{\mathbf{R}}_h^*\|_{L^2_\alpha(\Omega)} \leq C\log^{m'}(1/h)\|\mathbf{curl}\,\mathbf{v}_h^*\|_{L^2_\alpha(\Omega)}. \tag{5.13}$$

Here the exponent m' is in general smaller than the exponent m in Theorem 3.1 since the number of subdomains $\Omega_k \subset \Omega\backslash D_s$ is smaller than N_0. Combining (5.8) with (5.10), we obtain the final decomposition

$$\mathbf{v}_h = \nabla(\tilde{p}_h^s + \tilde{p}_h^*) + \mathbf{r}_h(\tilde{\mathbf{w}}_h^s + \tilde{\mathbf{w}}_h^*) + (\tilde{\mathbf{R}}_h^s + \tilde{\mathbf{R}}_h^*) = \nabla p_h + \mathbf{r}_h\mathbf{w}_h + \mathbf{R}_h \quad \text{on } \Omega \tag{5.14}$$

with $p_h \in Z_h(\Omega)$, $\mathbf{w}_h \in (Z_h(\Omega))^3$ and $\mathbf{R}_h \in V_h(\Omega)$.

Step 3. Verify the weighted stability of the discrete Helmholtz decomposition.

Let G be a polyhedron in $\Im_s^* \cup \tilde{\Im}$. By the stability of the discrete harmonic extensions and the "face" lemma and "edge" lemma , we can deduce that (see the proof of Theorem 3.1 in [18])

$$\|\tilde{p}_h^s\|_{1,G} \lesssim \log(1/h)\sum_{\bar{\Omega}_i\cap\bar{G}\neq\emptyset}\|p_h^s\|_{1,\Omega_i}$$

and

$$\|\tilde{\mathbf{w}}_h^s\|_{1,G} \lesssim \log(1/h)\sum_{\bar{\Omega}_i\cap\bar{G}\neq\emptyset}\|\mathbf{w}_h^s\|_{1,\Omega_i}$$

for any $G \in \Im_s^* \cup \tilde{\Im}$. Let α_G and β_G denote the restrictions of the coefficients on the polyhedron G. Then, by the above two inequalities, together with (5.5)-(5.6), we get

$$\beta_G^{\frac{1}{2}}\|\tilde{p}_h^s\|_{1,G} \lesssim \log(1/h)\sum_{\bar{\Omega}_i\cap\bar{G}\neq\emptyset}\beta_i^{\frac{1}{2}}\|p_h^s\|_{1,\Omega_i} \lesssim \log^2(1/h)\sum_{\bar{\Omega}_i\cap\bar{G}\neq\emptyset}\beta_i^{\frac{1}{2}}\|\mathbf{v}_h\|_{\mathbf{curl},\Omega_i} \tag{5.15}$$

and

$$\begin{aligned}\|\alpha_G^{\frac{1}{2}}\nabla\tilde{\mathbf{w}}_h^s\|_{0,G} + \|\beta_G^{\frac{1}{2}}\tilde{\mathbf{w}}_h^s\|_{0,G} &\lesssim \log(1/h)\sum_{\bar{\Omega}_i\cap\bar{G}\neq\emptyset}\alpha_i^{\frac{1}{2}}\|\mathbf{w}_h^s\|_{1,\Omega_i}\\ &\lesssim \log^2(1/h)\sum_{\bar{\Omega}_i\cap\bar{G}\neq\emptyset}\|\alpha_i^{\frac{1}{2}}\mathbf{curl}\mathbf{v}_h\|_{0,\Omega_i}.\end{aligned} \tag{5.16}$$

Here we have used the assumptions $\alpha_G \leq \alpha_i$ (which was explained after **Proposition 5.1**), $\beta_G \leq \beta_i$ (see **Assumption 3.1**) and $\beta_i \leq \alpha_i$ (see **Assumption 3.2**). Similarly, we have

$$h^{-1}\|\alpha_G^{\frac{1}{2}}\tilde{\mathbf{R}}_h^s\|_{0,G} \lesssim h^{-1}\log(1/h)\sum_{\bar{\Omega}_i\cap\bar{G}\neq\emptyset}\|\alpha_i^{\frac{1}{2}}\mathbf{R}_h^s\|_{0,\Omega_i} \lesssim \log^2(1/h)\sum_{\bar{\Omega}_i\cap\bar{G}\neq\emptyset}\|\alpha_i^{\frac{1}{2}}\mathbf{curl}\mathbf{v}_h\|_{0,\Omega_i}. \tag{5.17}$$

Combining (5.8) with (5.15)-(5.17), yields

$$\begin{aligned}\|\mathbf{v}_h^*\|_{H^*(\mathbf{curl},\ \Omega)} &\lesssim \|\mathbf{v}_h\|_{H^*(\mathbf{curl},\ \Omega)} + \|\nabla\tilde{p}_h^s + \mathbf{r}_h\tilde{\mathbf{w}}_h^s + \tilde{\mathbf{R}}_h^s\|_{H^*(\mathbf{curl},\ \Omega)}\\ &\lesssim \log^2(1/h)\|\mathbf{v}_h\|_{H^*(\mathbf{curl},\ \Omega)}.\end{aligned}$$

and

$$\|\mathbf{curl}\,\mathbf{v}_h^*\|_{L^2_\alpha(\Omega)} \lesssim \|\mathbf{curl}\,\mathbf{v}_h\|_{L^2_\alpha(\Omega)} + \|\mathbf{curl}(\mathbf{r}_h\tilde{\mathbf{w}}_h^s + \tilde{\mathbf{R}}_h^s)\|_{L^2_\alpha(\Omega)}$$

$$\lesssim \log^2(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{L^2_\alpha(\Omega)}$$

Substituting this into (5.11)-(5.13) and using the estimates (5.5)-(5.7), we can obtain the estimates

$$\|p_h\|_{H^1_\beta(\Omega)} \le C\log^{m'+2}(1/h)\|\mathbf{v}_h\|_{H^*(\mathbf{curl},\,\Omega)}, \tag{5.18}$$

$$\|\mathbf{w}_h\|_{H^1_*(\Omega)} \le C\log^{m'+2}(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{L^2_\alpha(\Omega)} \tag{5.19}$$

and

$$h^{-1}\|\mathbf{R}_h\|_{L^2_\alpha(\Omega)} \le C\log^{m'+2}(1/h)\|\mathbf{curl}\,\mathbf{v}_h\|_{L^2_\alpha(\Omega)}. \tag{5.20}$$

This implies that Theorem 3.2 is valid with $\hat{m} = m' + 2$. If there is no weird vertex, we have $V_h^*(\Omega) = V_h(\Omega)$ and $D_s = \emptyset$. Then **Step 1** in the above proof is unnecessary, and the exponent $m' + 2$ in the final estimates should be replaced with m'. ♯

Remark 5.1 *In the proof of Theorem 3.2, we cannot use Theorem 3.1 of [18] to get the L^2 stability as in Section 4 since Γ_r may contain some edges of Ω_r. Because of this, we have to use Theorem 4.1 of [18] and require* **Assumption 3.2** *to be satisfied.*

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